QUANTUM INVARIANTS, MODULAR FORMS, AND LATTICE POINTS II

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ABSTRACT. We study the SU(2) Witten–Reshetikhin–Turaev invariant for the Seifert fibered homology spheres with M-exceptional fibers. We show that the WRT invariant can be written in terms of (differential of) the Eichler integrals of modular forms with weight $1/2$ and $3/2$. By use of nearly modular property of the Eichler integrals we shall obtain asymptotic expansions of the WRT invariant in the large-N limit. We further reveal that the number of the gauge equivalent classes of flat connections, which dominate the asymptotics of the WRT invariant in $N \to \infty$, is related to the number of integral lattice points inside the M-dimensional tetrahedron.

1. INTRODUCTION

The Witten invariant for the 3-manifold $\mathcal{M}$ is defined by the Chern–Simons path integral as [58] (see also Ref. 3)

$$Z_k(\mathcal{M}) = \int \exp\left(2\pi\,\mathrm{i}\,k\ \mathrm{CS}(A)\right)\ \mathcal{D}A \tag{1.1}$$

where $k \in \mathbb{Z}$, and $\mathrm{CS}(A)$ is the Chern–Simons functional

$$\mathrm{CS}(A) = \frac{1}{8\pi^2}\int_{\mathcal{M}} \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}\,A \wedge A \wedge A\right) \tag{1.2}$$

In a limit $k \to \infty$ of the Witten invariant $Z_k(\mathcal{M})$, we may apply the saddle point method. As the saddle point of the Chern–Simons functional (1.2) denotes the flat connection

$$\mathrm{d}A + A \wedge A = 0 \tag{1.3}$$

the asymptotics of the partition function becomes a sum of the Chern–Simons invariants, and it is expected to be [3, 11, 58]

$$Z_k(\mathcal{M}) \sim \frac{1}{2}\,\mathrm{e}^{-\frac{3}{4}\pi\mathrm{i}(1+b^1)} \sum_{\alpha} (k+2)^{(\dim H^1 - \dim H^0)/2} \times \sqrt{T_\alpha}\,\mathrm{e}^{-2\pi\mathrm{i}(I_\alpha/4 + \dim H^0/8)}\,\mathrm{e}^{2\pi\mathrm{i}(k+2)\,\mathrm{CS}(A_\alpha)} \tag{1.4}$$

Here the sum of α denotes a gauge equivalent class of flat connections, and T_α and I_α respectively denote the Reidemeister torsion and the spectral flow. The first Betti number is b^1, and H^i is the cohomology space.

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To study the asymptotic behavior of the Witten invariant rigorously, we need explicit expression of the invariant. Alternative and combinatorial definition of this quantum invariant was given by Reshetikhin and Turaev [46] (see also Ref. 27). We denote $\tau_N(\mathcal{M})$ as the Witten–Reshetikhin–Turaev (WRT) invariant, which is related to the Witten invariant $Z_k(\mathcal{M})$ by

$$Z_k(\mathcal{M}) = \frac{\tau_{k+2}\,(\mathcal{M})}{\tau_{k+2}\,(S^2 \times S^1)} \tag{1.5}$$

and we have

$$\tau_N\left(S^3\right) = 1$$

$$\tau_N\left(S^2 \times S^1\right) = \sqrt{\frac{N}{2}}\,\frac{1}{\sin(\pi/N)}$$

Using this definition of the WRT invariant, asymptotic behavior of the WRT invariants for certain 3-manifolds has been extensively studied [14, 30, 31, 48–52].

Several years ago, Lawrence and Zagier found a connection between the WRT invariant and modular form [32]. They showed that the WRT invariant $\tau_N(\mathcal{M})$ for the Poincaré homology sphere $\mathcal{M} = \Sigma(2,3,5)$ can be regarded as a limiting value of the Eichler integral of vector modular form with weight $3/2$. Thanks to this correspondence, the exact asymptotic expansion of the WRT invariant in the large-N limit can be computed, and topological invariants such as the Chern–Simons invariant and the Reidemeister torsion can be interpreted from the viewpoint of modular forms. Meanwhile it has been established that this remarkable structure of the quantum invariants holds for the WRT invariants for 3-manifolds such as the Brieskorn homology spheres [19], 4-exceptional fibered Seifert homology spheres [20], and the spherical Seifert manifolds [21]. Also established is a connection between the Eichler integrals of vector modular forms with weight $1/2$ and the special values of the colored Jones polynomial for the torus knot $\mathcal{T}_{s,t}$ [24] (see also Refs. 16, 17, 25, 60) and the torus link $\mathcal{T}_{2,2m}$ [18].

One of the benefit of the quantum invariant/modular form correspondence is an observation that a limiting value of the Ramanujan mock theta functions [45] in $q \to \mathrm{e}^{2\pi \mathrm{i}/N}$ from outside a unit circle coincides with the WRT invariants for the spherical Seifert manifolds [22]. This fact opens up a new insight to modular forms and the Ramanujan mock theta functions, and we can expect that further studies on the quantum invariant/modular form correspondence should be fruitful.

In this article, as a continuation of Refs. 19, 20, we study an exact asymptotic expansion of the WRT invariant $\tau_N(\mathcal{M})$ for the M-exceptional fibered Seifert integral homology sphere $\mathcal{M} = \Sigma(p_1, p_2, \dots, p_M)$, where p_j are pairwise coprime positive integers. By use of modular forms with half-integral weight, we derive an asymptotic expansion in $N \to \infty$ number theoretically.

This paper is organized as follows. In Section 2 we review the construction of the WRT invariant for the Seifert fibered homology spheres following Ref. 31. An explicit form of the WRT invariant is given. Also discussed is an integral expression of the invariant. In Section 3 we introduce a family of vector modular forms with half-integral weight. We define the Eichler integrals thereof, and study the nearly modular property of a limiting value of the Eichler integrals. By use of this quasi modular transformation property, we compute the asymptotic expansion of the WRT invariant in the large-N limit in Section 4. We shall see that the invariant is a limiting value of the holomorphic function [13]. We study a contribution of dominating terms in the large-N limit in detail, and reveal a relationship with the number of the integral lattice points inside the higher dimensional tetrahedron. Also given is an explicit relationship between the Casson invariant and the first non-trivial coefficient of the Ehrhart polynomial. In Section 5 we give some results based on numerical computations. We compare the exact value of the WRT invariant with our asymptotic formula. The last section is devoted to conclusion and discussions.

2. WRT Invariant for Seifert Integral Homology Sphere

Following Ref. 31, we compute the WRT invariant $\tau_N(\mathcal{M})$ for the Seifert fibered integral homology sphere with M-exceptional fibers $\mathcal{M} = \Sigma(\vec{p}) = \Sigma(p_1, p_2, \dots, p_M)$ where p_j are pairwise coprime positive integers. Hereafter we use $\vec{p}$ as M-tuple

$$\vec{p} = (p_1, p_2, \dots, p_M)$$

The Seifert fibered integral homology sphere $\Sigma(\vec{p})$ has a rational surgery description as Fig. 1 (see, *e.g.*, Refs. 42, 54, 55), and the fundamental group has a presentation

$$\pi_1\left(\Sigma(\vec{p})\right) = \left\langle x_1, x_2, \dots, x_M, h \;\middle|\; \begin{array}{c} h \text{ is center} \\ x_j^{\,p_j} = h^{-q_j} \text{ for } 1 \leq j \leq M \\ x_1\, x_2 \cdots x_M = 1 \end{array} \right\rangle \tag{2.1}$$

Here $q_j \in \mathbb{Z}$ is coprime to p_j, and we have a constraint so that the fundamental group (2.1) gives the homology sphere;

$$P \sum_{j=1}^{M} \frac{q_j}{p_j} = 1 \tag{2.2}$$

Here and hereafter we use

$$P = P(\vec{p}) = \prod_{j=1}^{M} p_j \tag{2.3}$$

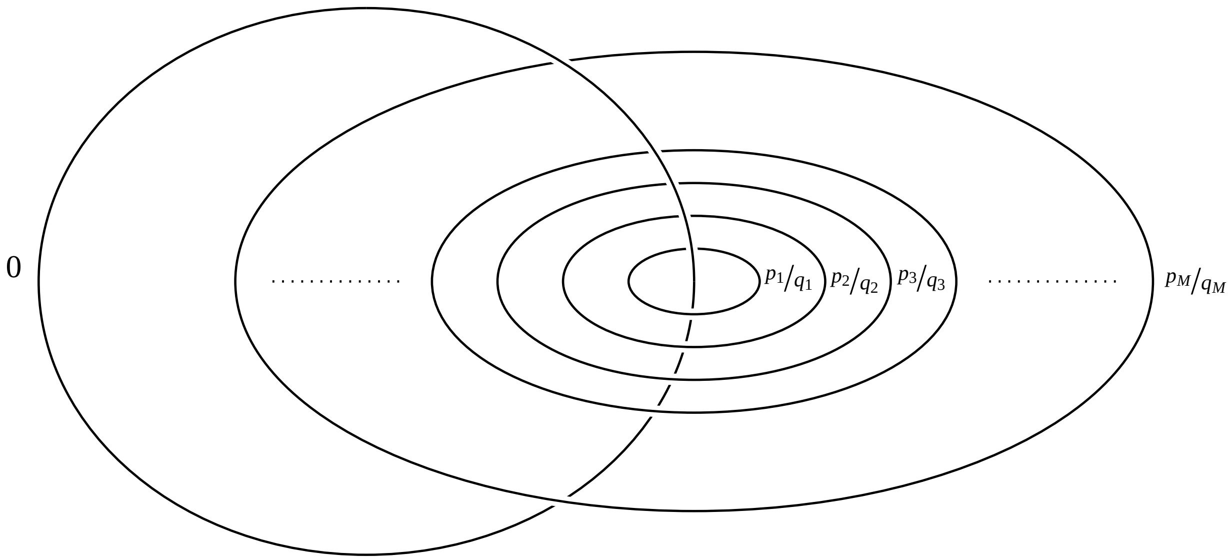

Figure 1: Surgery description of the Seifert homology sphere $\Sigma(p_1, \ldots, p_M)$

When the 3-manifold $\mathcal{M}$ is constructed by the rational surgeries p_j/q_j on the j-th component of n-component link $\mathcal{L}$, it was shown [26, 46] that the SU(2) WRT invariant $\tau_N(\mathcal{M})$ is given by

$$\tau_N(\mathcal{M}) = \mathrm{e}^{\frac{\pi \mathrm{i}}{4} \frac{N-2}{N} \left(\sum_{j=1}^{n} \Phi(U^{(p_j, q_j)}) - 3\,\mathrm{sign}(\mathbf{L})\right)} \sum_{k_1, \ldots, k_n = 1}^{N-1} J_{k_1, \ldots, k_n}(\mathcal{L}) \prod_{j=1}^{n} \rho(U^{(p_j, q_j)})_{k_j, 1} \tag{2.4}$$

Here the surgery data p_j/q_j is encoded by an $SL(2; \mathbb{Z})$ matrix

$$U^{(p_j, q_j)} = \begin{pmatrix} p_j & r_j \\ q_j & s_j \end{pmatrix}$$

The Rademacher Φ-function $\Phi(U)$ is defined by [43]

$$\Phi\left(\begin{pmatrix} p & r \\ q & s \end{pmatrix}\right) = \begin{cases} \dfrac{p+s}{q} - 12\, s(p, q) & \text{for } q \neq 0 \\ \dfrac{r}{s} & \text{for } q = 0 \end{cases} \tag{2.5}$$

where $s(b, a)$ denotes the Dedekind sum (A.1). An $n \times n$ matrix $\mathbf{L}$ is a linking matrix

$$\mathbf{L}_{j,k} = \mathrm{lk}(j, k) + \frac{p_j}{q_j} \cdot \delta_{j,k} \tag{2.6}$$

where $\mathrm{lk}(j, k)$ denotes the linking number of the j- and k-th components of link $\mathcal{L}$, and $\mathrm{sign}(\mathbf{L})$ denotes a signature of $\mathbf{L}$, *i.e.*, the difference between the number of positive and negative eigenvalues of $\mathbf{L}$. The polynomial $J_{k_1, \ldots, k_n}(\mathcal{L})$ denotes the colored Jone polynomial for link $\mathcal{L}$ with

color k_j for the j-th component link, and $\rho(U^{(p,q)})$ is a representation ρ of $PSL(2;\mathbb{Z})$ defined by

$$\rho(U^{(p,q)})_{a,b} = -\mathrm{i}\frac{\operatorname{sign}(q)}{\sqrt{2N|q|}}\,\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}\Phi(U^{(p,q)})}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{2Nq}sb^2}\sum_{\substack{\gamma \mod 2Nq \\ \gamma=a \mod 2N}}\mathrm{e}^{\frac{\pi\mathrm{i}}{2Nq}p\gamma^2}\left(\mathrm{e}^{\frac{\pi\mathrm{i}}{Nq}\gamma b}-\mathrm{e}^{-\frac{\pi\mathrm{i}}{Nq}\gamma b}\right) \tag{2.7}$$

for $1 \le a, b \le N-1$ [26]. This representation is constructed from

$$\begin{aligned}\rho(S)_{a,b} &= \sqrt{\frac{2}{N}}\sin\left(\frac{a\,b}{N}\pi\right)\\ \rho(T)_{a,b} &= \mathrm{e}^{\frac{\pi\mathrm{i}}{2N}a^2-\frac{\pi\mathrm{i}}{4}}\,\delta_{a,b}\end{aligned} \tag{2.8}$$

with

$$S = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \qquad\qquad T = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \tag{2.9}$$

satisfying

$$S^2 = (S\,T)^3 = 1$$

Based on the fact that the Seifert fibered manifold $\Sigma(\vec{p})$ has a surgery description as in Fig. 1, we have the following result.

Proposition 1 ([31]). *For the Seifert fibered integral homology sphere with M-exceptional fibers $\mathcal{M} = \Sigma(\vec{p})$, the WRT invariant is given by*

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\left(\frac{\phi(\vec{p})}{4}-\frac{1}{2}\right)}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}-1\right)\tau_N(\mathcal{M}) = \frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{2\sqrt{2PN}}\sum_{\substack{n=0\\ N\nmid n}}^{2PN-1}\mathrm{e}^{-\frac{1}{2PN}n^2\pi\mathrm{i}}\,\frac{\prod_{j=1}^{M}\left(\mathrm{e}^{\frac{n}{Np_j}\pi\mathrm{i}}-\mathrm{e}^{-\frac{n}{Np_j}\pi\mathrm{i}}\right)}{\left(\mathrm{e}^{\frac{n}{N}\pi\mathrm{i}}-\mathrm{e}^{-\frac{n}{N}\pi\mathrm{i}}\right)^{M-2}} \tag{2.10}$$

Here $\phi(\vec{p})$ is defined by

$$\phi(\vec{p}) = 3 - \frac{1}{P} + 12\sum_{j=1}^{M} s\left(\frac{P}{p_j}, p_j\right) \tag{2.11}$$

where $s(b,a)$ denotes the Dedekind sum (A.1).

Outline of Proof. We use the surgery formula (2.4), in which the the colored Jones polynomial for a variant of Hopf link $\mathcal{L}$ depicted in Fig. 1 is given by

$$J_{k_0,k_1,\dots,k_M}(\mathcal{L}) = \frac{1}{\sin(\pi/N)}\,\frac{\prod_{j=1}^{M}\sin\left(\frac{k_0 k_j}{N}\pi\right)}{\left[\sin\left(\frac{k_0}{N}\pi\right)\right]^{M-1}}$$

Here k_0 is a color of component which has a linking number 1 with any other components of $\mathcal{L}$. After some computations using the Gauss sum reciprocity formula (A.7), we get

$$\begin{aligned}
&\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\left(\frac{\phi(\vec{p})}{4}-\frac{1}{2}\right)} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}-1\right) \tau_N(\mathcal{M}) \\
&\qquad = \frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{2\sqrt{2PN}} \sum_{k_0=1}^{N-1} \sum_{n_j \mod p_j} \frac{1}{\left(\mathrm{e}^{\frac{k_0}{N}\pi\mathrm{i}}-\mathrm{e}^{-\frac{k_0}{N}\pi\mathrm{i}}\right)^{M-2}} \\
&\qquad\qquad \times \prod_{j=1}^{M} \mathrm{e}^{-\frac{q_j}{p_j}\frac{(k_0+2Nn_j)^2}{2N}} \left(\mathrm{e}^{\frac{k_0+2Nn_j}{Np_j}\pi\mathrm{i}}-\mathrm{e}^{-\frac{k_0+2Nn_j}{Np_j}\pi\mathrm{i}}\right)
\end{aligned} \tag{2.12}$$

We see that the summand in (2.12) is invariant under

- $k_0 \to k_0 + 2N$ and $n_j \to n_j - 1$ for all j,
- $n_j \to n_j + p_j$.

Using that p_j are pairwise coprime, we can then rewrite the multi-sum of (2.12) into a single-sum of (2.10). □

The WRT invariant can be rewritten in the integral form as follows;

Proposition 2 ([31]).

$$\begin{aligned}
&\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\left(\frac{\phi(\vec{p})}{4}-\frac{1}{2}\right)} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}-1\right) \tau_N(\mathcal{M}) \\
&\qquad = \frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{2\sqrt{2PN}} \left(-2\pi\mathrm{i}\sum_{m=0}^{2P-1} \operatorname*{Res}_{z=mN} \frac{g(z)}{1-\mathrm{e}^{-2\pi\mathrm{i}z}} + \int_C g(z)\,\mathrm{d}z\right)
\end{aligned} \tag{2.13}$$

where

$$g(z) = \mathrm{e}^{-\frac{z^2}{2PN}\pi\mathrm{i}} \frac{\prod_{j=1}^{M}\left(\mathrm{e}^{\frac{z}{Np_j}\pi\mathrm{i}}-\mathrm{e}^{-\frac{z}{Np_j}\pi\mathrm{i}}\right)}{\left(\mathrm{e}^{\frac{z}{N}\pi\mathrm{i}}-\mathrm{e}^{-\frac{z}{N}\pi\mathrm{i}}\right)^{M-2}} \tag{2.14}$$

and the integration path C passes the origin from $(-1+\mathrm{i})\infty$ *to* $(1-\mathrm{i})\infty$.

Outline of Proof. Key identity is

$$\Theta_M(x) = \Theta_M(x-N)\,\mathrm{e}^{4\pi\mathrm{i}Px} + 2\pi\mathrm{i}\sum_{m=0}^{2P-1}\operatorname*{Res}_{z=mN} h_M(z,x) + \sum_{\substack{n=0 \\ N\nmid n}}^{2PN-1} f_M(n,x) \tag{2.15}$$

Here the function $\Theta_M(x)$ is defined by

$$\Theta_M(x) = \int_C h_M(z,x)\,\mathrm{d}z$$

where

$$h_M(z,x) = \mathrm{e}^{-\frac{z^2}{2PN}\pi\mathrm{i}+2x\frac{z}{N}\pi\mathrm{i}} \frac{1}{\left(\mathrm{e}^{\frac{z}{N}\pi\mathrm{i}} - \mathrm{e}^{-\frac{z}{N}\pi\mathrm{i}}\right)^{M-2}} \cdot \frac{1}{1-\mathrm{e}^{-2\pi\mathrm{i}z}} \equiv \frac{f_M(z,x)}{1-\mathrm{e}^{-2\pi\mathrm{i}z}}$$

As the function $g(z)$ is a linear combination of $f_M(x,z)$, we obtain the expression (2.13). □

In view of (2.13), we can decompose the invariant as

$$\tau_N(\mathcal{M}) = \tau_N^{\mathrm{res}}(\mathcal{M}) + \tau_N^{\mathrm{int}}(\mathcal{M}) \tag{2.16}$$

where $\tau_N^{\mathrm{res}}(\mathcal{M})$ and $\tau_N^{\mathrm{int}}(\mathcal{M})$ respectively denote contributions from the residue terms and the integral term in (2.13). It was identified in Ref. 31 that the residue part $\tau_N^{\mathrm{res}}(\mathcal{M})$ is the contribution from irreducible flat connections while the integral term $\tau_N^{\mathrm{int}}(\mathcal{M})$ is the trivial connection contribution. The trivial connection contribution is related to the Ohtsuki series [40], and we have as follows;

Proposition 3. *In the limit $N \to \infty$, the trivial connection contribution has the asymptotic expansion as*

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\left(\frac{\phi(\vec{p})}{4}-\frac{1}{2}\right)} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} - 1\right) \tau_N^{\mathrm{int}}(\mathcal{M}) \simeq \sum_{k=0}^{\infty} \frac{T_{\vec{p}}(k)}{k!} \left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^k \tag{2.17}$$

where the T-series is given by

$$\frac{\prod_{j=1}^{M} \sinh\left(\frac{P}{p_j}\,x\right)}{\left[\sinh\left(Px\right)\right]^{M-2}} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{T_{\vec{p}}(k)}{(2\,k)!}\,x^{2k} \tag{2.18}$$

We will discuss later the relationship with the Ohtsuki series.

Similar integral with $\tau_N^{\mathrm{int}}(\mathcal{M})$ in the case of the three exceptional fibers $M = 3$ appeared in studies [1, 36] of the Ramanujan mock theta functions, which still remain to be mysterious and fascinating topics. This suggests the remarkable fact that the Ramanujan mock theta functions [45] are related to the WRT invariant for the Seifert fibered manifolds. See Refs. 21–23 for detail.

The dominating term of the WRT invariant in a limit $N \to \infty$ follows from the irreducible flat connection contributions $\tau_N^{\mathrm{res}}(\mathcal{M})$ as was expected from the saddle point approximation (1.4). In terms of the Witten partition function $Z_k(\mathcal{M})$, the asymptotic expansion of the invariant is given by

$$Z_{k-2}(\mathcal{M}) \simeq \sum_{a=0}^{M-3} N^{M-3-a}\, Z_{N-2}^{(a)}(\mathcal{M}) + \text{trivial connection contribution}$$

and the dominating term $Z_{N-2}^{(0)}(\mathcal{M})$ in $N \to \infty$ can be computed as follows when we use an identity

$$\sin(\pi\,z) = \pi\,z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

Proposition 4. *In the large N limit, the Witten partition function $Z_N(\mathcal{M})$ for the M-exceptional fibered Seifert homology sphere $\mathcal{M} = \Sigma(\vec{p})$ is dominated by*

$$Z_{N-2}(\mathcal{M}) \sim N^{M-3} \frac{2^{M-2}}{(M-2)!\,\sqrt{P}} \mathrm{e}^{-\frac{\phi(\vec{p})}{2N}\pi\mathrm{i}} \mathrm{e}^{-\frac{2M-3}{4}\pi\mathrm{i}} \times \sum_{m=0}^{2P-1} (-1)^{mM} B_{M-2}\left(\frac{m}{2\,P}\right) \mathrm{e}^{-\frac{m^2}{2P}\pi\mathrm{i}N} \left[\prod_{j=1}^{M} \sin\left(\frac{m}{p_j}\pi\right)\right] \tag{2.19}$$

where $B_k(x)$ is the k-th Bernoulli polynomial (A.8).

Among a sum of $2\,P$ terms in the right hand side of (2.19), we can classify the summation by the Chern–Simons invariant, which corresponds to an exponential factor $-\frac{m^2}{4\,P} \mod 1$ as will be discussed later.

3. Modular Forms and Eichler Integral

We introduce the vector modular forms with half-integral weight which play a crucial role in analysis of the WRT invariants for the Seifert fibered homology spheres.

3.1 Vector Modular Forms with Half-Integral Weight

We set M-tuple

$$\vec{\ell} = (\ell_1, \ldots, \ell_M) \tag{3.1}$$

where ℓ_j are integers satisfying $0 < \ell_j < p_j$. As before we assume that p_j are pairwise coprime positive integers. For M-tuple $\vec{\ell}$, we define the periodic function $\chi_{2P}^{\vec{\ell}}(n)$ with modulus $2\,P$ by

$$\chi_{2P}^{\vec{\ell}}(n) = \begin{cases} -\prod_{j=1}^{M} \varepsilon_j, & \text{if } n = P\left(1 + \sum_{j=1}^{M} \varepsilon_j \frac{\ell_j}{p_j}\right) \mod 2\,P \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

where $\varepsilon_j = \pm 1$ for $\forall j$. We see that $\chi_{2P}^{\vec{\ell}}(n)$ is even (resp. odd) when M is even (resp. odd),

$$\chi_{2P}^{\vec{\ell}}(-n) = (-1)^M \chi_{2P}^{\vec{\ell}}(n) \tag{3.3}$$

and that it has a mean value zero,

$$\sum_{n=0}^{2P-1} \chi_{2P}^{\vec{\ell}}(n) = 0 \tag{3.4}$$

We define an involution σ_j on M-tuple $\vec{\ell}$ by

$$\sigma_j(\vec{\ell}) = (\ell_1, \ldots, \ell_{j-1}, p_j - \ell_j, \ell_{j+1}, \ldots, \ell_M) \tag{3.5}$$

for $1 \leq j \leq M$. As we have

$$\chi_{2P}^{\sigma_i \sigma_j(\vec{\ell})}(n) = \chi_{2P}^{\vec{\ell}}(n) \tag{3.6}$$

for $1 \leq i, j \leq M$, the number of the independent periodic functions $\chi_{2P}^{\vec{\ell}}(n)$ for given $\vec{p}$ is

$$D = D(\vec{p}) = \frac{1}{2^{M-1}} \prod_{j=1}^{M} \left(p_j - 1\right) \tag{3.7}$$

We note that

$$\chi_{2P}^{\vec{\ell}}(n + P) = -\chi_{2P}^{\sigma_j(\vec{\ell})}(n) \tag{3.8}$$

for $1 \leq j \leq M$.

We set

$$q = \exp(2\pi \mathrm{i} \tau) \tag{3.9}$$

where τ is in the upper half plane, $\tau \in \mathbb{H}$. By use of the periodic functions (3.2) we define the q-series by

$$\Phi_{\vec{p}}^{\vec{\ell}}(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} n^{\mathrm{m}_2(M)} \chi_{2P}^{\vec{\ell}}(n)\, q^{\frac{n^2}{4P}} \tag{3.10}$$

where we mean

$$\mathrm{m}_2(M) = \frac{1 - (-1)^M}{2} = M \mod 2 = \begin{cases} 0 & \text{when } M \text{ is even} \\ 1 & \text{when } M \text{ is odd} \end{cases} \tag{3.11}$$

The q-series $\Phi_{\vec{p}}^{\vec{\ell}}(\tau)$ is proved to be a vector modular form with half-integral weight. The T-transformation is trivial, and by use of the Poisson summation formula,

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} t n} f(t)\, \mathrm{d}t \tag{3.12}$$

we obtain the transformation formula under the S-transformation as follows.

Proposition 5. *The q-series $\Phi_{\vec{p}}^{\vec{\ell}}(\tau)$ is a vector modular form with weight* $3/2$ *(resp.* $1/2$*) when M is even (resp. odd). Under the S- and T-transformations* (2.9) *we have*

$$\Phi_{\vec{p}}^{\vec{\ell}}(\tau) = \left(\frac{\mathrm{i}}{\tau}\right)^{\frac{3}{2} - \mathrm{m}_2(M)} \sum_{\vec{\ell}'} \mathbf{S}_{\vec{\ell}'}^{\vec{\ell}}\, \Phi_{\vec{p}}^{\vec{\ell}'}(-1/\tau) \tag{3.13}$$

$$\Phi_{\vec{p}}^{\vec{\ell}}(\tau + 1) = \mathbf{T}^{\vec{\ell}}\, \Phi_{\vec{p}}^{\vec{\ell}}(\tau) \tag{3.14}$$

Here a sum of $\vec{\ell}'$ runs over D-dimensional space (3.7), *and matrix elements of $D \times D$ matrices* $\mathbf{S}$ *and* $\mathbf{T}$ *are respectively given by*

$$\mathbf{S}_{\vec{\ell}'}^{\vec{\ell}} = \frac{2^M \, \mathrm{i}^{M-\mathrm{m}_2(M)}}{\sqrt{2\,P}} \, (-1)^{P\left(1+\sum_{j=1}^M \frac{\ell_j+\ell'_j}{p_j}\right)+P\sum_j \sum_{k\neq j} \frac{\ell_j \ell'_k}{p_j p_k}} \prod_{j=1}^M \sin\left(P \, \frac{\ell_j \, \ell'_j}{{p_j}^2} \, \pi\right) \tag{3.15}$$

$$\mathbf{T}^{\vec{\ell}} = \exp\left(\frac{P}{2}\left(1 + \sum_{j=1}^M \frac{\ell_j}{p_j}\right)^2 \pi \, \mathrm{i}\right) \tag{3.16}$$

Our vector modular form $\Phi_{\vec{p}}^{\vec{\ell}}(\tau)$ may be a generalization of modular forms which appear as the character of the affine Lie algebra $\widehat{su(2)}$ (a case of $M = 1$, and $\Psi_P^{(a)}(\tau)$ defined below) and as the minimal Virasoro model (a case of $M = 2$) up to the power of the Dedekind η function.

For our later use, we introduce other families of vector modular forms. We define the even periodic functions by

$$\theta_{2P}^{(a)}(n) = \begin{cases} 1 & \text{for } n = \pm a \mod 2\,P \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

for $0 \leq a \leq P$, and the odd periodic function by

$$\psi_{2P}^{(a)}(n) = \begin{cases} \pm 1 & \text{for } n \equiv \pm a \mod 2\,P \\ 0 & \text{otherwise} \end{cases} \tag{3.18}$$

for $0 < a < P$. We then define two families of q-series by

$$\Theta_P^{(a)}(\tau) = \frac{1}{2} \sum_{n\in\mathbb{Z}} \theta_{2P}^{(a)}(n) \, q^{\frac{n^2}{4P}} \tag{3.19}$$

$$\Psi_P^{(a)}(\tau) = \frac{1}{2} \sum_{n\in\mathbb{Z}} n \, \psi_{2P}^{(a)}(n) \, q^{\frac{n^2}{4\,P}} \tag{3.20}$$

These families are also vector modular forms with half-integral weight. Namely we see that $\Theta_P^{(a)}(\tau)$ is a vector modular form with weight $1/2$ satisfying

$$\Theta_P^{(a)}(\tau) = \sqrt{\frac{\mathrm{i}}{\tau}} \, \sum_{b=0}^P \mathbf{N}_b^a \, \Theta_P^{(b)}(-1/\tau) \tag{3.21}$$

$$\Theta_P^{(a)}(\tau+1) = \exp\left(\frac{a^2}{2\,P} \, \pi \, \mathrm{i}\right) \, \Theta_P^{(a)}(\tau) \tag{3.22}$$

where $\mathbf{N}$ is $(P+1)\times(P+1)$ matrix defined by

$$\mathbf{N}^a_b = \begin{cases} \dfrac{1}{\sqrt{2\,P}} & \text{for } a=0 \\ \sqrt{\dfrac{2}{P}}\cos\left(\dfrac{a\,b}{P}\,\pi\right) & \text{for } a\neq 0, P \\ \dfrac{1}{\sqrt{2P}}\cos\left(b\,\pi\right) & \text{for } a=P \end{cases} \tag{3.23}$$

The vector modular form $\Psi^{(a)}_P(\tau)$ with weight $3/2$ fulfills the following transformation formulae;

$$\Psi^{(a)}_P(\tau) = \left(\frac{\mathrm{i}}{\tau}\right)^{3/2}\sum_{b=1}^{P-1}\mathbf{M}^a_b\,\Psi^{(b)}_P(-1/\tau) \tag{3.24}$$

$$\Psi^{(a)}_P(\tau+1) = \exp\left(\frac{a^2}{2\,P}\,\pi\,\mathrm{i}\right)\,\Psi^{(a)}_P(\tau) \tag{3.25}$$

where $\mathbf{M}$ is a $(P-1)\times(P-1)$ matrix defined by

$$\mathbf{M}^a_b = \sqrt{\frac{2}{P}}\,\sin\left(\frac{a\,b}{P}\,\pi\right) \tag{3.26}$$

3.2 Eichler Integrals

The Eichler integral is originally defined for modular forms with integral weight ≥ 2 (see, *e.g.*, Ref. 29). In our cases, the vector modular forms $\Phi^{\vec{\ell}}_{\vec{p}}(\tau)$ have a half-integral weight, so we follow a method of Refs. 32, 60 to define a variant of the Eichler integrals.

We define the Eichler integrals $\widetilde{\Phi}^{\vec{\ell}}_{\vec{p}}(\tau)$ of the vector modular form $\Phi^{\vec{\ell}}_{\vec{p}}(\tau)$ by

$$\widetilde{\Phi}^{\vec{\ell}}_{\vec{p}}(\tau) = \sum_{n=0}^{\infty} n^{1-\mathrm{m}_2(M)}\,\chi^{\vec{\ell}}_{2P}(n)\,q^{\frac{n^2}{4P}} \tag{3.27}$$

This can be regarded as a “half-derivative” (resp. “half-integral”) of the modular form $\Phi^{\vec{\ell}}_{\vec{p}}(\tau)$ with respect to τ when M is even (resp. odd). When M is odd, the q-series $\widetilde{\Phi}^{\vec{\ell}}_{\vec{p}}(\tau)$ might be called the false theta function à la Rogers [47].

Proposition 6. *We assume N_1 and N_2 are coprime integers, and $N_1 > 0$. Limiting values of the Eichler integrals $\widetilde{\Phi}^{\vec{\ell}}_{\vec{p}}(\tau)$ in $\tau \to N_2/N_1$ are given as follows;*

$$\widetilde{\Phi}^{\vec{\ell}}_{\vec{p}}(N_2/N_1) = -\,(P\,N_1)^{1-\mathrm{m}_2(M)}\sum_{k=0}^{2PN_1}\chi^{\vec{\ell}}_{2P}(k)\,\mathrm{e}^{\frac{N_2}{N_1}\frac{k^2}{2P}\pi\,\mathrm{i}}\,B_{2-\mathrm{m}_2(M)}\left(\frac{k}{2\,P\,N_1}\right) \tag{3.28}$$

where $B_k(x)$ is the k-th Bernoulli polynomial.

We note that

$$\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(\tau+1) = \mathbf{T}^{\vec{\ell}}\, \widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(\tau)$$

from which we have for $N \in \mathbb{Z}$

$$\begin{aligned} \widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(N) &= \left(\mathbf{T}^{\vec{\ell}}\right)^N \widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(0) \\ &= -\left(\mathbf{T}^{\vec{\ell}}\right)^N P^{1-\mathrm{m}_2(M)} \sum_{k=0}^{2P} \chi_{2P}^{\vec{\ell}}(k)\, B_{2-\mathrm{m}_2(M)}\left(\frac{k}{2P}\right) \end{aligned}$$

A proof follows straightforwardly when we use the following lemma [32] (see also Ref. 41).

Lemma 7. *Let $C_f(n)$ is a periodic function with modulus f and mean value zero. Then we have as $t \searrow 0$*

$$\sum_{n=1}^{\infty} C_f(n)\, \mathrm{e}^{-n^2 t} \simeq \sum_{n=0}^{\infty} L\left(-2n, C_f\right) \frac{(-t)^n}{n!} \tag{3.29}$$

$$\sum_{n=1}^{\infty} n\, C_f(n)\, \mathrm{e}^{-n^2 t} \simeq \sum_{n=0}^{\infty} L\left(-2n-1, C_f\right) \frac{(-t)^n}{n!} \tag{3.30}$$

where the L-function is

$$\begin{aligned} L\left(s, C_f\right) &= \sum_{n=1}^{\infty} \frac{C_f(n)}{n^s} \\ &= f^{-s} \sum_{k=1}^{f} C_f(k)\, \zeta\left(s, \frac{k}{f}\right) \end{aligned} \tag{3.31}$$

Note that the Hurwitz zeta function $\zeta(s,z)$ defined by

$$\zeta(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s} \tag{3.32}$$

has an analytic continuation for $k \in \mathbb{Z}_{>0}$ as

$$\zeta(1-k, z) = -\frac{B_k(z)}{k} \tag{3.33}$$

Proposition 8. *The limiting values $\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(\alpha)$ of the Eichler integrals with $\alpha \in \mathbb{Q}$ satisfy a nearly modular property. In a limit $N \to \infty$, we have the transformation formula as an asymptotic*

expansion as follows;

$$\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(1/N) + \left(\frac{N}{\mathrm{i}}\right)^{\frac{3}{2}-\mathrm{m}_2(M)} \sum_{\vec{\ell}'} \mathbf{S}_{\vec{\ell}'}^{\vec{\ell}}\, \widetilde{\Phi}_{\vec{p}}^{\vec{\ell}'}(-N) \simeq \sum_{k=0}^{\infty} \frac{L\left(-2\,k-1+\mathrm{m}_2(M), \chi_{2P}^{\vec{\ell}}\right)}{k!} \left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^k \tag{3.34}$$

Here $N \in \mathbb{Z}$, and a sum of M-tuples $\vec{\ell}'$ runs over D-dimensional space.

Proof. We introduce another variant of the Eichler integral by

$$\widehat{\Phi}_{\vec{p}}^{\vec{\ell}}(z) = \begin{cases} -\sqrt{\dfrac{P\,\mathrm{i}}{2\,\pi^2}} \displaystyle\int_{\overline{z}}^{\infty} \frac{\Phi_{\vec{p}}^{\vec{\ell}}(\tau)}{(\tau-z)^{3/2}}\,\mathrm{d}\tau, & \text{when } M \text{ is even} \\[2ex] \dfrac{1}{\sqrt{2\,P\,\mathrm{i}}} \displaystyle\int_{\overline{z}}^{\infty} \frac{\Phi_{\vec{p}}^{\vec{\ell}}(\tau)}{\sqrt{\tau-z}}\,\mathrm{d}\tau, & \text{when } M \text{ is odd} \end{cases} \tag{3.35}$$

where we assume that z is in the lower half-plane, $z \in \mathbb{H}^-$, and $\overline{z}$ denotes a complex conjugate of z. By use of the S-transformation (3.13), we have

$$\widehat{\Phi}_{\vec{p}}^{\vec{\ell}}(z) + \left(\frac{1}{\mathrm{i}\,z}\right)^{\frac{3}{2}-\mathrm{m}_2(M)} \sum_{\vec{\ell}'} \mathbf{S}_{\vec{\ell}'}^{\vec{\ell}}\, \widehat{\Phi}_{\vec{p}}^{\vec{\ell}'}(-1/z) = r_{\Phi_{\vec{p}}^{\vec{\ell}}}(z;0) \tag{3.36}$$

Here $r_{\Phi_{\vec{p}}^{\vec{\ell}}}(z;\alpha)$ is an analogue of the period function defined by

$$r_{\Phi_{\vec{p}}^{\vec{\ell}}}(z;\alpha) = \begin{cases} -\sqrt{\dfrac{P\,\mathrm{i}}{2\,\pi^2}} \displaystyle\int_{\alpha}^{\infty} \frac{\Phi_{\vec{p}}^{\vec{\ell}}(\tau)}{(\tau-z)^{3/2}}\,\mathrm{d}\tau, & \text{when } M \text{ is even} \\[2ex] \dfrac{1}{\sqrt{2\,P\,\mathrm{i}}} \displaystyle\int_{\alpha}^{\infty} \frac{\Phi_{\vec{p}}^{\vec{\ell}}(\tau)}{\sqrt{\tau-z}}\,\mathrm{d}\tau, & \text{when } M \text{ is odd} \end{cases} \tag{3.37}$$

where $\alpha \in \mathbb{Q}$. We find that $\widehat{\Phi}_{\vec{p}}^{\vec{\ell}}(z)$ takes the same limiting value with that of $\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(\tau)$ in a limit $z, \tau \to \alpha \in \mathbb{Q}$;

$$\left.\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(\tau)\right|_{\tau\to\alpha} = \left.\widehat{\Phi}_{\vec{p}}^{\vec{\ell}}(z)\right|_{z\to\alpha} \tag{3.38}$$

The right hand side of (3.34) arises from an asymptotic expansion of $r_{\Phi_{\vec{p}}^{\vec{\ell}}}(z;0)$, and we obtain (3.34). □

For our later use, we study differentials of the Eichler integral $\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(\tau)$, *i.e.*, "fractional derivatives" of the vector modular form $\Phi_{\vec{p}}^{\vec{\ell}}(\tau)$. From the definition (3.27) of the Eichler integral, we

have for $b \in \mathbb{Z}_{\geq 0}$

$$\left(\frac{2\,P}{\pi\,\mathrm{i}}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{b}\,\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(\tau)=\sum_{n=0}^{\infty}n^{2b+1-\mathrm{m}_2(M)}\,\chi_{2P}^{\vec{\ell}}(n)\,q^{\frac{n^2}{4P}} \tag{3.39}$$

By the same computation with (3.28), we have the following.

Proposition 9. *The limiting values of* fractional *derivative of the vector modular forms* $\Phi_{\vec{p}}^{\vec{\ell}}(\tau)$ *are given by*

$$\left.\left(\frac{2\,P}{\pi\,\mathrm{i}}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{b}\,\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(\tau)\right|_{\tau\to\frac{N_2}{N_1}} \\ =-\frac{(2\,P\,N_1)^{2b+1-\mathrm{m}_2(M)}}{2\,b+2-\mathrm{m}_2(M)}\sum_{n=1}^{2PN_1}\chi_{2P}^{\vec{\ell}}(n)\,\mathrm{e}^{\frac{N_2}{N_1}\frac{n^2}{2P}\pi\mathrm{i}}\,B_{2b+2-\mathrm{m}_2(M)}\left(\frac{n}{2\,P\,N_1}\right) \tag{3.40}$$

where $N_1>0$ *and* N_2 *are coprime integers.*

The nearly modular property (3.34) of the Eichler integral gives the following asymptotic expansion of (3.40).

Proposition 10. *In the limit* $N\to\infty$*, we have the following asymptotic expansion;*

$$N^{2b+1-\mathrm{m}_2(M)}\sum_{n=1}^{2PN}\chi_{2P}^{\vec{\ell}}(n)\,\mathrm{e}^{\frac{n^2}{2PN}\pi\mathrm{i}}\,B_{2b+2-\mathrm{m}_2(M)}\left(\frac{n}{2\,P\,N}\right) \\ \simeq\frac{-1}{\mathrm{i}^{\frac{3}{2}-\mathrm{m}_2(M)}}\sum_{j=0}^{b}N^{b+j+\frac{3}{2}-\mathrm{m}_2(M)}\left(\frac{\mathrm{i}}{2\,P\,\pi}\right)^{b-j}K_{b,\mathrm{m}_2(M)}^{(j)} \\ \times\frac{2\,b+2-\mathrm{m}_2(M)}{2\,j+2-\mathrm{m}_2(M)}\sum_{\vec{\ell}'}\mathbf{S}_{\vec{\ell}'}^{\vec{\ell}}\left[\sum_{n=1}^{2P}\chi_{2P}^{\vec{\ell}'}(n)\,B_{2j+2-\mathrm{m}_2(M)}\left(\frac{n}{2\,P}\right)\right]\mathrm{e}^{-\frac{P}{2}\left(1+\sum_j\frac{\ell'_j}{p_j}\right)^2\pi\mathrm{i}N} \\ -\frac{2\,b+2-\mathrm{m}_2(M)}{(2\,P)^{2b+1-\mathrm{m}_2(M)}}\sum_{k=0}^{\infty}\frac{L\left(-2\,k-2\,b-1+\mathrm{m}_2(M),\chi_{2P}^{\vec{\ell}}\right)}{k!}\left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^{k} \tag{3.41}$$

where the sum of M-tuples $\vec{\ell}'$ *runs over D-dimensional space, and we have*

$$K_{b,x}^{(j)}=\binom{b}{j}\prod_{k=0}^{b-j-1}\left(\frac{1}{2}+b-x-k\right) \tag{3.42}$$

Proof. We differentiate (3.36) with respect to z, and then take a limit $z \to 1/N$. We get

$$\frac{\mathrm{d}^b}{\mathrm{d}z^b}\widehat{\Phi}^{\vec{\ell}}_{\vec{p}}(z)\bigg|_{z\to\frac{1}{N}} + \frac{1}{\mathrm{i}^{\frac{1}{2}+\mathrm{m}_2(M)}}\sum_{\vec{\ell}'}\mathbf{S}^{\vec{\ell}}_{\vec{\ell}'}\left(w^2\,\frac{\mathrm{d}}{\mathrm{d}w}\right)^b\, w^{\frac{3}{2}-\mathrm{m}_2(M)}\,\widehat{\Phi}^{\vec{\ell}'}_{\vec{p}}(w)\bigg|_{w\to -N} \simeq \left(\frac{\pi\,\mathrm{i}}{2\,P}\right)^b\sum_{k=0}^{\infty}\frac{L\left(-2\,k-2\,b-1+\mathrm{m}_2(M),\chi^{\vec{\ell}}_{2P}\right)}{k!}\left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^k \tag{3.43}$$

In the left hand side, we use

$$\left(w^2\,\frac{\mathrm{d}}{\mathrm{d}w}\right)^n = \sum_{m=1}^{n} A_n^{(m)}\,w^{n+m}\,\frac{\mathrm{d}^m}{\mathrm{d}w^m}$$

where $A_n^{(m)}$ is the Lah number defined by

$$A_n^{(m)} = \frac{n!}{m!}\binom{n-1}{m-1}$$

This denotes the number of partitions of $\{1,2,\ldots,n\}$ into m *lists* (a "list" denotes an ordered subset) [56, 57], and satisfies the recursion relation

$$A_{n+1}^{(m)} = A_n^{(m-1)} + (n+m)\,A_n^{(m)}$$

Thus we have

$$\left(w^2\,\frac{\mathrm{d}}{\mathrm{d}w}\right)^b\, w^{\frac{3}{2}-\mathrm{m}_2(M)}\,\widehat{\Phi}^{\vec{\ell}}_{\vec{p}}(w) = \sum_{j=0}^{b} w^{\frac{3}{2}-\mathrm{m}_2(M)+b+j}\,K^{(j)}_{b,\mathrm{m}_2(M)}\,\frac{\mathrm{d}^j\widehat{\Phi}^{\vec{\ell}}_{\vec{p}}(w)}{\mathrm{d}w^j}$$

Here the K-number is computed as

$$K^{(j)}_{b,x} = \sum_{k=j}^{b} A_b^{(k)}\binom{k}{j}\frac{\Gamma\left(\frac{5}{2}-x\right)}{\Gamma\left(\frac{5}{2}-x-k+j\right)}$$

which reduces to (3.42) applying (A.14) and (A.17).

As we have (3.38), we get (3.41) with a help of (3.40). □

In the same method, we have formulae of the asymptotic expansions concerning to the periodic functions with mean value zero.

Corollary 11. *We assume that $C_{2P}(n)$ is an odd or even periodic function with modulus $2\,P$, and that it satisfies*

- *a mean value zero condition,*

$$\sum_{n=0}^{2P-1} C_{2P}(n) = 0$$

- $C_{2P}(0) = 0$.

In the limit $N \to \infty$, we have the following asymptotic expansions;

- *$C_{2P}(n)$ is odd;*

$$\begin{aligned} N^{2b} \sum_{n=1}^{2PN} C_{2P}(n)\, \mathrm{e}^{\frac{n^2}{2PN}\pi \mathrm{i}}\, B_{2b+1}\left(\frac{n}{2\,P\,N}\right) \\ \simeq \frac{-2}{\mathrm{i}^{\frac{1}{2}}} \sum_{j=0}^{b} N^{b+j+\frac{1}{2}} \left(\frac{\mathrm{i}}{2\,P\,\pi}\right)^{b-j} K_{b,1}^{(j)} \frac{2\,b+1}{2\,j+1} \sum_{a=1}^{P-1} C_{2P}(a) \sum_{c=1}^{P-1} \mathbf{M}_c^a B_{2j+1}\left(\frac{c}{2\,P}\right) \mathrm{e}^{-N\frac{c^2}{2P}\pi \mathrm{i}} \\ - \frac{2\,b+1}{(2\,P)^{2b}} \sum_{k=0}^{\infty} \frac{L\,(-2\,k-2\,b, C_{2P})}{k!} \left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^k \end{aligned} \tag{3.44}$$

- *$C_{2P}(n)$ is even;*

$$\begin{aligned} N^{2b+1} \sum_{n=1}^{2PN} C_{2P}(n)\, \mathrm{e}^{\frac{n^2}{2PN}\pi \mathrm{i}}\, B_{2b+2}\left(\frac{n}{2\,P\,N}\right) \\ \simeq \frac{-1}{\mathrm{i}^{\frac{3}{2}}} \sum_{j=0}^{b} N^{b+j+\frac{3}{2}} \left(\frac{\mathrm{i}}{2\,P\,\pi}\right)^{b-j} K_{b,0}^{(j)} \frac{b+1}{j+1} \sum_{a=1}^{P} C_{2P}(a) \\ \times \sum_{c=0}^{P} \mathbf{N}_c^a \,(2-\delta_{c,0}-\delta_{c,P})\, B_{2j+2}\left(\frac{c}{2\,P}\right) \mathrm{e}^{-N\frac{c^2}{2P}\pi \mathrm{i}} \\ - \frac{2\,b+2}{(2\,P)^{2b+1}} \sum_{k=0}^{\infty} \frac{L\,(-2\,k-2\,b-1, C_{2P})}{k!} \left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^k \end{aligned} \tag{3.45}$$

Proof. We use that

$$C_{2P}(n) = \begin{cases} \displaystyle\sum_{a=1}^{P-1} C_{2P}(a)\, \psi_{2P}^{(a)}(n) & \text{when } C_{2P}(n) \text{ is odd} \\ \displaystyle\sum_{a=1}^{P} C_{2P}(a)\, \theta_{2P}^{(a)}(n) & \text{when } C_{2P}(n) \text{ is even} \end{cases}$$

In both cases, due to a condition $C_{2P}(0) = 0$, we see that periodic functions which appear in the modular transformation formula such as (3.43) have a mean value zero. Applying Lemma 7, we obtain asymptotic expansions. □

4. Exact Asymptotic Expansion of the WRT invariants

4.1 Exact Asymptotic Expansion

As a preparation to obtain the exact asymptotic expansion of the WRT invariant $\tau_N(\mathcal{M})$ for the M-exceptional fibered Seifert homology sphere $\mathcal{M} = \Sigma(\vec{p})$ by use of the vector modular form $\Phi_{\vec{p}}^{\vec{\ell}}(\tau)$ defined in previous section, we give q-series identities at the root of unity related to the Bernoulli polynomials.

Lemma 12. *We set ω_N as the N-th primitive root of unity;*

$$\omega_N = \exp\left(\frac{2\pi \mathrm{i}}{N}\right)$$

and assume that a and k are positive integers satisfying $0 \leq a \leq N-1$. *We have*

$$\sum_{c=1}^{N-1} \frac{\omega_N^{(a+1)c}}{\left(1-\omega_N^c\right)^k} = \frac{(-1)^k}{(k-1)!} \sum_{j=1}^{k} \frac{S_k^{(j)}}{j} \left(B_j(1) - N^j B_j\left(\frac{a+1}{N}\right)\right) \tag{4.1}$$

where $S_k^{(j)}$ is the Stirling number of the first kind (A.14).

Proof. We follow a method in Ref. 2.

We define the function $P(t;k,a)$ by

$$P(t;k,a) = \sum_{c=1}^{N-1} \frac{\omega_N^{(a+1)c}}{\left(1-\omega_N^c \,\mathrm{e}^t\right)^k} \tag{4.2}$$

where a and k are positive integers. The function $P(t;k=1,a)$ is computed as follows;

$$\begin{aligned} P(t;k=1,a) &= \sum_{n=0}^{\infty} \sum_{c=1}^{N-1} \omega_N^{(a+1)c+nc} \,\mathrm{e}^{nt} \\ &= -\frac{1}{1-\mathrm{e}^t} + N\,\mathrm{e}^{-(a+1)t} \frac{\mathrm{e}^{Nt}}{1-\mathrm{e}^{Nt}} \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \left(B_{m+1}(0) - N^{m+1} B_{m+1}\left(1-\frac{a+1}{N}\right)\right) t^m \end{aligned} \tag{4.3}$$

Here we have used

$$\sum_{c=1}^{N-1} \omega_N^{nc} = \begin{cases} N-1, & \text{if } N \mid n, \\ -1, & \text{otherwise.} \end{cases}$$

We further introduce the function $Q(t;k,c)$ by

$$Q(t;k,c) = \frac{1}{\left(1-\omega_N^c \,\mathrm{e}^t\right)^k}, \tag{4.4}$$

where k and c are positive integers. By definitions we have

$$P(t;k,a) = \sum_{c=1}^{N-1} \omega_N^{(a+1)c}\, Q(t;k,c) \tag{4.5}$$

The definition (4.4) indicates that the function $Q(t;k,c)$ satisfies a differential-difference equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(t;k,c) = k\left(Q(t;k+1,c) - Q(t;k,c)\right)$$

We can check by induction that the function $Q(t;k,c)$ can be written in terms of $Q(t;k=1,c)$ as

$$Q(t;k,c) = \frac{(-1)^{k+1}}{(k-1)!} \sum_{m=0}^{k-1} (-1)^m S_k^{(m+1)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} Q(t;1,c) \tag{4.6}$$

From (4.5) we find that $P(t;k,a)$ is solved as

$$P(t;k,a) = \frac{(-1)^{k+1}}{(k-1)!} \sum_{m=0}^{k-1} (-1)^m S_k^{(m+1)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} P(t;1,a) \tag{4.7}$$

Substituting (4.3) for the above solution, we complete the proof. □

Using the arithmetic identity (4.1), we can rewrite the WRT invariant (2.10) in terms of the Bernoulli polynomials.

Proposition 13. *The WRT invariant for the M-exceptional fibered Seifert integral homology sphere $\mathcal{M} = \Sigma(\vec{p})$, which was computed as in* (2.10)*, is written in terms of the Bernoulli polynomials as*

$$\begin{aligned}
\mathrm{e}^{\frac{2\pi \mathrm{i}}{N}\left(\frac{\phi(\vec{p})}{4}-\frac{1}{2}\right)} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} - 1\right) \tau_N(\mathcal{M}) = -\frac{1}{2}\, \frac{1}{(M-3)!} \sum_{j=0}^{M-3} (-N)^j\, \frac{S_{M-2}^{(j+1)}}{j+1} \\
\times \sum_{n=0}^{2PN-1} \chi_{2P}^{\vec{E}}(n)\, \mathrm{e}^{\frac{1}{2PN}(n+P(M-3))^2 \pi \mathrm{i}}\, B_{j+1}\left(\frac{1}{N}\left\lfloor \frac{n}{2P} \right\rfloor\right) \\
+ \frac{(-1)^M}{(M-4)!} \sum_{a=1}^{\infty} \sum_{b=0}^{a-1} \sum_{\vec{\eta}(a)}{}' \sum_{n=0}^{N-1} \sum_{j=1}^{M-3} \frac{S_{M-3}^{(j)}}{j}\, \mathrm{e}^{\pi\mathrm{i} \frac{2P}{N}\left(n-b-\frac{M-4}{2}+\frac{1}{2}\sum_{i=1}^{M} \frac{\eta_i}{p_i}\right)^2} \\
\times \left(N^{j-1} B_j\left(\frac{n+1}{N}\right) - \frac{1}{N} B_j(1) \right)
\end{aligned} \tag{4.8}$$

where for our brevity we have used M-tuple

$$\vec{E} = (\underbrace{1,1,\dots,1}_{M}) \tag{4.9}$$

and $\sum_{\vec{\eta}(a)}'$ *is a signed sum*

$$\sum_{\vec{\eta}(a)}' \cdots = \sum_{\substack{\vec{\eta}\in\{\pm 1\}^M s.t. \\ 2a-1<\sum_{j=1}^M \frac{\eta_j}{p_j}<2a+1}} \left[\prod_{j=1}^M \eta_j\right] \cdots$$

We remark that the second term including a sum of a in (4.8) vanishes when

$$\sum_{j=1}^M \frac{1}{p_j} < 1$$

Even when $\sum_{j=1}^M \frac{1}{p_j} > 1$, the second term is a finite sum. It is well known that the sum of inverse of prime numbers, $\sum_{p:\ \text{prime}} \frac{1}{p}$ diverges, although the sum up to the $10,000$-th prime numbers is still $2.709258\cdots$.

Proof of Prof. 13. We first study a case of $\sum_j \frac{1}{p_j} < 1$. In this case, we have

$$-z^P \prod_{j=1}^M \left(z^{\frac{P}{p_j}} - z^{-\frac{P}{p_j}}\right) = \sum_{m=0}^{2P-1} \chi_{2P}^{\vec{E}}(m)\, z^m \tag{4.10}$$

where the periodic function $\chi_{2P}^{\vec{E}}(m)$ is defined in (3.2). Using this identity in (2.10), we have

$$\begin{aligned}
&\mathrm{e}^{\frac{2\pi \mathrm{i}}{N}\left(\frac{\phi(\vec{p})}{4}-\frac{1}{2}\right)} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} - 1\right) \tau_N(\mathcal{M}) \\
&= -\frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{2\sqrt{2PN}} \sum_{m=0}^{2P-1}\sum_{k=1}^{N-1} \chi_{2P}^{\vec{E}}(m)\, \frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{2PN}(m+(M-3)P)^2}}{\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}k}-1\right)^{M-2}} \sum_{j=0}^{2P-1} \mathrm{e}^{-\pi\mathrm{i}\frac{N}{2P}\left(j+\frac{k-m-(M-3)P}{N}\right)^2} \\
&= \frac{-1}{2N}\sum_{m=0}^{2P-1}\sum_{n=0}^{N-1} \chi_{2P}^{\vec{E}}(m)\, \mathrm{e}^{\frac{\pi\mathrm{i}}{2PN}(2Pn-m-(M-3)P)^2} \sum_{k=1}^{N-1} \frac{\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}kn}}{\left(\mathrm{e}^{\frac{2\pi\mathrm{i}k}{N}}-1\right)^{M-2}} \\
&= \frac{-1}{2}\,\frac{1}{(M-3)!} \sum_{m=0}^{2P-1}\sum_{n=0}^{N-1} \chi_{2P}^{\vec{E}}(m)\, \mathrm{e}^{\frac{\pi\mathrm{i}}{2PN}(2Pn+m+(M-3)P)^2} \\
&\qquad \times \sum_{j=0}^{M-3} \frac{S_{M-2}^{(j+1)}}{j+1} \left(\frac{1}{N} B_{j+1}(1) - N^j B_{j+1}\left(1-\frac{n}{N}\right)\right)
\end{aligned}$$

Here in the first equality we have used (4.10), and decomposed a sum of n by setting $n = N\, j + k$. We have then applied the Gauss sum reciprocity formula (A.7) in the second equality, and then used our formula (4.1) in the last equality. As we have

$$\sum_{n=0}^{2PN-1} \chi_{2P}^{\vec{\ell}}(n)\, \mathrm{e}^{\frac{\pi\mathrm{i}}{2PN}n^2} = 0 \tag{4.11}$$

due to (3.4), we obtain the first term of (4.8).

For other cases $\sum_j \frac{1}{p_j} > 1$, the generating function (4.10) of the periodic function $\chi_{2P}^{\vec{E}}(n)$ is replaced with

$$
\begin{aligned}
& - z^P \prod_{j=1}^{M} \left(z^{\frac{P}{p_j}} - z^{-\frac{P}{p_j}} \right) + \sum_{a=1}^{\infty} \sum_{\vec{\eta}(a)}{}' z^P \left(z^{aP} - z^{-aP} \right) \\
& \qquad \times \left(z^{P\left(\sum_j \frac{\eta_j}{p_j} - a\right)} + (-1)^{M+1} z^{-P\left(\sum_j \frac{\eta_j}{p_j} - a\right)} \right) = \sum_{n=0}^{2P-1} \chi_{2P}^{\vec{E}}(n)\, z^n
\end{aligned}
\tag{4.12}
$$

Thus comparing with a case of $\sum_j \frac{1}{p_j} < 1$, we need additional term τ_{add} defined by

$$
\begin{aligned}
\tau_{\mathrm{add}} = & \sum_{a=1}^{\infty} \sum_{b=0}^{a-1} \sum_{\vec{\eta}(a)}{}' \sum_{\substack{n=0 \\ N \nmid n}}^{2PN-1} \frac{\mathrm{e}^{\frac{\pi \mathrm{i}}{4}}}{2\sqrt{2\,P\,N}} \frac{\mathrm{e}^{-\frac{n^2}{2PN}\pi \mathrm{i}}}{\left(\mathrm{e}^{\frac{n}{N}\pi \mathrm{i}} - \mathrm{e}^{-\frac{n}{N}\pi \mathrm{i}} \right)^{M-3}} \\
& \qquad \times \left(\mathrm{e}^{\pi \mathrm{i} \frac{n}{N}\left(\sum_j \frac{\eta_j}{p_j} - 2b - 1\right)} + (-1)^{M+1} \mathrm{e}^{-\pi \mathrm{i} \frac{n}{N}\left(\sum_j \frac{\eta_j}{p_j} - 2b - 1\right)} \right)
\end{aligned}
\tag{4.13}
$$

We decompose a sum of n by setting $n = N\, j + k$, and apply the Gauss sum reciprocity formula (A.7). After some computations, we obtain

$$
\begin{aligned}
\tau_{\mathrm{add}} &= \frac{1}{2\,N} \sum_{a=1}^{\infty} \sum_{b=0}^{a-1} \sum_{\vec{\eta}(a)}{}' \sum_{k=1}^{N-1} \sum_{n=0}^{N-1} \frac{\mathrm{e}^{\pi \mathrm{i} \frac{2P}{N}\left(n - \frac{M-2}{2} + \frac{1}{2}\sum_j \frac{\eta_j}{p_j}\right)^2}}{\left(\mathrm{e}^{\frac{k}{N}\pi \mathrm{i}} - \mathrm{e}^{-\frac{k}{N}\pi \mathrm{i}} \right)^{M-3}} \\
& \qquad \times \left(\mathrm{e}^{\pi \mathrm{i} \frac{k}{N}(M-3-2b-2n)} + (-1)^{M+1} \mathrm{e}^{\pi \mathrm{i} \frac{k}{N}(2b+2n-M+3)} \right) \\
&= \frac{1}{N} \sum_{a=1}^{\infty} \sum_{b=0}^{a-1} \sum_{\vec{\eta}(a)}{}' \sum_{n=0}^{N-1} \mathrm{e}^{\pi \mathrm{i} \frac{2P}{N}\left(n - \frac{M-2}{2} + \frac{1}{2}\sum_j \frac{\eta_j}{p_j}\right)^2} \sum_{k=1}^{N-1} \frac{\mathrm{e}^{2\pi \mathrm{i} \frac{k}{N}(b+n)}}{\left(1 - \mathrm{e}^{2\pi \mathrm{i} \frac{k}{N}}\right)^{M-3}}
\end{aligned}
$$

where we have used the fact that the sum (4.1) is real. Substituting (4.1) for the above, we find that τ_{add} gives the second term of r.h.s. of (4.8), and thus we complete the proof. □

We now aim to relate this expression with limiting values of the Eichler integrals (3.28) and differentials (3.40) thereof. For our convention, we introduce an analogue of the Bernoulli polynomial defined by

$$
f_m^M(x) = \sum_{k=m}^{M} \frac{1}{k} S_M^{(k)} \binom{k}{m} \left(x + \frac{M}{2} \right)^{k-m}
\tag{4.14}
$$

where $M, m \in \mathbb{Z}$ satisfying $M \geq m > 0$. For a case of $m = 0$, we set

$$
f_0^M(x) = \sum_{k=1}^{M} \frac{S_M^{(k)}}{k} \left(x + \frac{M}{2} \right)^k
\tag{4.15}
$$

Some of explicit forms of the polynomials $f_m^M(x)$ are given below;

$$
\begin{aligned}
f_M^M(x) &= \frac{1}{M} \\
f_{M-1}^M(x) &= x \\
f_{M-2}^M(x) &= \frac{1}{M}\binom{M}{2}\left(x^2 - \frac{M}{12}\right) \\
f_{M-3}^M(x) &= \frac{1}{M}\binom{M}{3}\left(x^3 - \frac{M}{4}\,x\right) \\
f_{M-4}^M(x) &= \frac{1}{M}\binom{M}{4}\left(x^4 - \frac{M}{2}\,x^2 + \frac{1}{240}\,M\,(5\,M+2)\right)
\end{aligned}
$$

Lemma 14. *Let the polynomial* $f_m^M(x)$ *be defined by* (4.14). *Then the polynomial* $f_{M-k\neq 0}^M(x)$ *is even (resp. odd) when* k *is even (resp. odd).*

Proof. We introduce the generating function of the polynomials $f_M^m(x)$ by

$$F_M(x,y) = \sum_{m=0}^{M} m\, f_m^M(x)\, y^{m-1} \tag{4.16}$$

Recalling the generating function (A.14) of the Stirling number of the first kind, we get

$$F_M(x,y) = \prod_{j=1}^{M-1}\left(y + x + \frac{M}{2} - j\right) \tag{4.17}$$

which shows that $F_M(x,y)$ is a polynomial of $x+y$. Furthermore $F_M(x,y)$ becomes an odd (resp. even) polynomial of $x+y$ when M is even (resp. odd). Then we can conclude that the polynomial $f_{M-k}^M(x)$ is even (resp. odd) if k is even (resp. odd). □

By use of the generating function (4.17), we obtain the following differential equation and recursion relation of $f_m^M(x)$;

$$\frac{\mathrm{d}}{\mathrm{d}x} f_m^M(x) = (m+1)\, f_{m+1}^M(x) \tag{4.18}$$

$$f_j^{M+1}\left(x - \frac{1}{2}\right) = \left(x - \frac{M}{2}\right) f_j^M(x) + \frac{j-1}{j}\, f_{j-1}^M(x) \tag{4.19}$$

We can rewrite the WRT invariant in Prop. 13 in terms of these polynomials as follows.

Proposition 15. *The WRT invariant $\tau_N(\mathcal{M})$ for the M-exceptional fibered Seifert homology sphere $\mathcal{M} = \Sigma(\vec{p})$ is written as*

$$\begin{aligned}
& \mathrm{e}^{\frac{2\pi \mathrm{i}}{N}\left(\frac{\phi(\vec{p})}{4}-\frac{1}{2}\right)} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}-1\right) \tau_N(\mathcal{M}) \\
& \quad = -\frac{1}{2}\,\frac{1}{(M-3)!} \sum_{k=1}^{M-2} (-N)^{k-1} \sum_{n=0}^{2PN-1} \chi_{2P}^{\vec{E}}(n)\, f_k^{M-2}\left(\frac{n}{2\,P} - \left\lfloor \frac{n}{2\,P} \right\rfloor - \frac{1}{2}\right) \\
& \qquad \times \mathrm{e}^{\frac{1}{2PN}(n+P(M-3))^2 \pi \mathrm{i}}\, B_k\left(\frac{n+P\,(M-3)}{2\,P\,N}\right) \\
& \qquad + \frac{(-1)^M}{(M-4)!} \sum_{a=1}^{\infty} \sum_{b=0}^{a-1} {\sum_{\vec{\eta}(a)}}' \sum_{n=0}^{N-1} \mathrm{e}^{\pi \mathrm{i} \frac{2P}{N}\left(n-b-\frac{M-4}{2}+\frac{1}{2}\sum_j \frac{\eta_j}{p_j}\right)^2} \\
& \qquad \times \sum_{k=0}^{M-3} \Biggl(N^{k-1}\, f_k^{M-3}\left(b + \frac{1}{2} - \frac{1}{2}\sum_j \frac{\eta_j}{p_j}\right) B_k\left(\frac{2\,n - 2\,b - M + 4 + \sum_j \frac{\eta_j}{p_j}}{2\,N}\right) \\
& \qquad\qquad - \frac{1}{N}\, f_k^{M-3}\left(\frac{5-M}{2}\right) B_k(0) \Biggr)
\end{aligned} \tag{4.20}$$

Proof. To prove for a case of $\sum_j \frac{1}{p_j} < 1$, in which the second term in (4.20) vanishes, we only need to apply (A.13) to $B_{j+1}\left(\frac{1}{N}\left\lfloor\frac{n}{2P}\right\rfloor\right)$ in the first term of (4.8). As we have an identity

$$\sum_{j=1}^{M} \frac{S_M^{(j)}}{j}\, B_j(x+y)\, z^j = \sum_{k=0}^{M} f_k^M\left(y\,z - \frac{M}{2}\right) B_k(x)\, z^k \tag{4.21}$$

we obtain the required expression.

In the case of $\sum_j \frac{1}{p_j} > 1$, we need to evaluate the second term in (4.8), which we have set τ_{add} in the proof of Prop. 13. This term τ_{add} can be transformed into the above expression (4.20) when we apply (4.21). □

Theorem 16. *The exact asymptotic expansion of the WRT invariant $\tau_N(\mathcal{M})$ for the Seifert homology sphere $\mathcal{M} = \Sigma(\vec{p})$ with M-singular fibers in $N \to \infty$ is given as follows;*

$$
\begin{aligned}
&\mathrm{e}^{\frac{2\pi \mathrm{i}}{N}\left(\frac{\phi(\vec{p})}{4}-\frac{1}{2}\right)} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}-1\right) \tau_N(\mathcal{M}) \\
&\simeq \frac{1}{2}\,\frac{1}{(M-3)!}\,\frac{1}{\mathrm{i}^{\frac{3}{2}-\mathrm{m}_2(M)}} \sum_{j=0}^{\lfloor\frac{M-3}{2}\rfloor} N^{j+\lfloor\frac{M}{2}\rfloor-\frac{1}{2}} \left(\frac{\mathrm{i}}{2P\pi}\right)^{\lfloor\frac{M-3}{2}\rfloor-j} K^{(j)}_{\lfloor\frac{M-3}{2}\rfloor,\mathrm{m}_2(M)} \\
&\times \frac{1}{2j+2-\mathrm{m}_2(M)} \sum_{\vec{\ell}'} \mathbf{S}_{\vec{\ell}'}^{\sigma_1^{M-1}(\vec{E})} \left[\sum_{n=1}^{2P} \chi_{2P}^{\vec{\ell}'}(n)\, B_{2j+2-\mathrm{m}_2(M)}\left(\frac{n}{2P}\right)\right] \mathrm{e}^{-N\frac{P}{2}\left(1+\sum_j \frac{\ell'_j}{p_j}\right)^2 \pi\mathrm{i}} \\
&+ \frac{1}{(M-3)!}\,\frac{(-1)^M}{\mathrm{i}^{\frac{3}{2}}} \sum_{m=1}^{\lfloor\frac{M-3}{2}\rfloor} \sum_{a=0}^{P} \chi_{2P}^{\sigma_1^{M-1}(\vec{E})}(a)\, f_{2m}^{M-2}\left(\frac{a}{2P}-\frac{\mathrm{m}_2(M)}{2}\right) \\
&\times \sum_{j=1}^{m} N^{m+j-\frac{1}{2}} \left(\frac{\mathrm{i}}{2P\pi}\right)^{m-j} K^{(j-1)}_{m-1,0}\,\frac{m}{j} \sum_{c=0}^{P} \mathbf{N}_c^a\, \frac{2-\delta_{c,0}-\delta_{c,P}}{2}\, B_{2j}\left(\frac{c}{2P}\right) \mathrm{e}^{-N\frac{c^2}{2P}\pi\mathrm{i}} \\
&+ \frac{1}{(M-3)!}\,\frac{(-1)^{M-1}}{\mathrm{i}^{\frac{1}{2}}} \sum_{m=0}^{\lfloor\frac{M-4}{2}\rfloor} \sum_{a=1}^{P} \chi_{2P}^{\sigma_1^{M-1}(\vec{E})}(a)\, f_{2m+1}^{M-2}\left(\frac{a}{2P}-\frac{\mathrm{m}_2(M)}{2}\right) \\
&\times \sum_{j=0}^{m} N^{m+j+\frac{1}{2}} \left(\frac{\mathrm{i}}{2P\pi}\right)^{m-j} K^{(j)}_{m,1}\,\frac{2m+1}{2j+1} \sum_{c=1}^{P-1} \mathbf{M}_c^a\, B_{2j+1}\left(\frac{c}{2P}\right) \mathrm{e}^{-N\frac{c^2}{2P}\pi\mathrm{i}} \\
&+ \frac{(-1)^M}{(M-4)!} \sum_{a=1}^{\infty} \sum_{b=0}^{a-1} \sum_{\vec{\eta}(a)}{}' \Bigg\{ \sum_{m=1}^{\lfloor\frac{M-3}{2}\rfloor} f_{2m}^{M-3}\left(b+\frac{1}{2}-\frac{1}{2}\sum_i \frac{\eta_i}{p_i}\right) \frac{-1}{\mathrm{i}^{\frac{3}{2}}} \sum_{j=1}^{m} N^{m+j-\frac{1}{2}} \left(\frac{\mathrm{i}}{2P\pi}\right)^{m-j} \\
&\times K^{(j-1)}_{m-1,0}\,\frac{m}{j} \sum_{c=0}^{P} \mathbf{N}_c^{\sigma^{M-1}(P\sum_i \frac{\eta_i}{p_i}-P(2a-1))}\, \frac{2-\delta_{c,0}-\delta_{c,P}}{2}\, B_{2j}\left(\frac{c}{2P}\right) \mathrm{e}^{-N\frac{c^2}{2P}\pi\mathrm{i}} \\
&+ (-1)^{M-1} \sum_{m=0}^{\lfloor\frac{M-4}{2}\rfloor} f_{2m+1}^{M-3}\left(b+\frac{1}{2}-\frac{1}{2}\sum_i \frac{\eta_i}{p_i}\right), \frac{-1}{\mathrm{i}^{\frac{1}{2}}} \sum_{j=0}^{m} N^{m+j+\frac{1}{2}} \left(\frac{\mathrm{i}}{2P\pi}\right)^{m-j} \\
&\times K^{(j)}_{m,1}\,\frac{2m+1}{2j+1} \sum_{c=1}^{P-1} \mathbf{M}_c^{\sigma^{M-1}(P\sum_i \frac{\eta_i}{p_i}-P(2a-1))} B_{2j+1}\left(\frac{c}{2P}\right) \mathrm{e}^{-N\frac{c^2}{2P}\pi\mathrm{i}} \Bigg\} \\
&\qquad\qquad + \sum_{k=0}^{\infty} \frac{T_{\vec{p}}(k)}{k!} \left(\frac{\pi\,\mathrm{i}}{2PN}\right)^k
\end{aligned}
\tag{4.22}
$$

where we have used an involution σ on $x \in \mathbb{Z}_{2P}$ defined by

$$
\sigma(x) = P - x \mod 2P \tag{4.23}
$$

The coefficients $T_{\vec{p}}(k)$ in a tail *part are defined by*

$$
\begin{aligned}
T_{\vec{p}}(k) = \frac{(-1)^{M+1}}{2} \frac{(2\,P)^{2k}}{(M-3)!} \sum_{n=1}^{2P} \chi_{2P}^{\sigma_1^{M-1}(\vec{E})}(n) \sum_{j=1}^{M-2} (-1)^j \frac{j}{2\,k+j} B_{2k+j}\left(\frac{n}{2\,P}\right) \\
\times f_j^{M-2}\left(\frac{n+(M-1)\,P}{2\,P} - \left\lfloor \frac{n+(M-1)\,P}{2\,P} \right\rfloor - \frac{1}{2}\right) \\
- \frac{(2\,P)^{2k}}{(M-4)!} \sum_{a=1}^{\infty} \sum_{b=0}^{a-1} \sum_{\vec{\eta}(a)}{}' \sum_{j=1}^{M-3} (-1)^{(M+1)(j+1)} \frac{j}{2\,k+j} \\
\times f_j^{M-3}\left(b + \frac{1}{2} - \frac{1}{2}\sum_i \frac{\eta_i}{p_i}\right) B_{2k+j}\left(\frac{1}{2\,P}\,\sigma^{M-1}\left(P\sum_i \frac{\eta_i}{p_i} - (2\,a-1)\,P\right)\right)
\end{aligned} \tag{4.24}
$$

Proof. We first study the case $\sum_j \frac{1}{p_j} < 1$, in which we only have the first term in (4.20). We shift the parameter n by $P\,(3-M)$, and we have

$$
\chi_{2P}^{\vec{E}}(n - P\,(M-3)) = (-1)^{1-\mathrm{m}_2(M)}\, \chi_{2P}^{\sigma_1^{M-1}(\vec{E})}(n)
$$

As we see that $\chi_{2P}^{\vec{\ell}}(n)\, f_k^{M-2}\left(\frac{n+(M-1)P}{2\,P} - \left\lfloor \frac{n+(M-1)P}{2\,P} \right\rfloor - \frac{1}{2}\right)$ is an even (resp. odd) periodic function when k is even (resp. odd), we can write by use of $f_M^M(x) = \frac{1}{M}$ that

$$
\begin{aligned}
& \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\left(\frac{\phi(\vec{p})}{4} - \frac{1}{2}\right)} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} - 1\right) \tau_N(\mathcal{M}) \\
&= -\frac{1}{2} \frac{N^{M-3}}{(M-2)!} \sum_{n=0}^{2PN-1} \chi_{2P}^{\sigma_1^{M-1}(\vec{E})}(n)\, \mathrm{e}^{\frac{n^2}{2PN}\pi\mathrm{i}}\, B_{M-2}\left(\frac{n}{2\,P\,N}\right) \\
&\quad - \frac{(-1)^M}{2} \frac{1}{(M-3)!} \sum_{c=1}^{\lfloor \frac{M-3}{2} \rfloor} N^{2c-1} \left[\sum_{n=0}^{2PN-1} \chi_{2P}^{\sigma_1^{M-1}(\vec{E})}(n)\, f_{2c}^{M-2}\left(\frac{n}{2\,P} - \frac{\mathrm{m}_2(M)}{2}\right) \mathrm{e}^{\frac{n^2}{2PN}\pi\mathrm{i}}\, B_{2c}\left(\frac{n}{2\,P\,N}\right)\right] \\
&\quad + \frac{(-1)^M}{2} \frac{1}{(M-3)!} \sum_{c=0}^{\lfloor \frac{M-4}{2} \rfloor} N^{2c} \left[\sum_{n=0}^{2PN-1} \chi_{2P}^{\sigma_1^{M-1}(\vec{E})}(n)\, f_{2c+1}^{M-2}\left(\frac{n}{2\,P} - \frac{\mathrm{m}_2(M)}{2}\right) \mathrm{e}^{\frac{n^2}{2PN}\pi\mathrm{i}}\, B_{2c+1}\left(\frac{n}{2\,P\,N}\right)\right]
\end{aligned}
$$

Generating function (4.10) proves

$$
\sum_{n=0}^{2P-1} \chi_{2P}^{\vec{E}}(n)\, g(n) = 0
$$

where $g(n)$ is an arbitrary polynomial of n of order at most $M-1$. So we find that $\chi_{2P}^{\vec{E}}(n)\, f_k^{M-2}\left(\frac{n}{2\,P} - \left\lfloor \frac{n}{2\,P} \right\rfloor - \frac{1}{2}\right)$ is a periodic function of n with mean value zero, and that the expression (4.20) can be identified with a limiting value of the Eichler integrals and their derivatives studied in the previous section. This proves that the WRT invariant is a limiting value of the holomorphic function of q in $q \to \mathrm{e}^{2\pi\mathrm{i}/N}$ [13]. Substituting both (3.44) and (3.45) for above expression, we get (4.22).

For the second term τ_{add} in (4.20), recalling the periodic functions $\psi_{2P}^{(a)}(n)$ and $\theta_{2P}^{(a)}(n)$ we can rewrite it into

$$\begin{aligned}\tau_{\text{add}} = \frac{1}{2}\frac{(-1)^M}{(M-4)!}\sum_{a=1}^{\infty}\sum_{b=0}^{a-1}\sum_{\vec{\eta}(a)}{}' \Bigg\{ &\sum_{m=0}^{\lfloor\frac{M-3}{2}\rfloor} N^{2m-1}\, f_{2m}^{M-3}\left(b+\frac{1}{2}-\frac{1}{2}\sum_i\frac{\eta_i}{p_i}\right) \\ &\times\sum_{n=0}^{2PN-1}\theta_{2P}^{(\sigma^{M-1}(P\sum_i\frac{\eta_i}{p_i}-(2a-1)P))}(n)\,\mathrm{e}^{\pi\mathrm{i}\frac{n^2}{2PN}}\,B_{2m}\left(\frac{n}{2PN}\right) \\ &+(-1)^{M-1}\sum_{m=0}^{\lfloor\frac{M-4}{2}\rfloor} N^{2m}\, f_{2m+1}^{M-3}\left(b+\frac{1}{2}-\frac{1}{2}\sum_i\frac{\eta_i}{p_i}\right) \\ &\times\sum_{n=0}^{2PN-1}\psi_{2P}^{(\sigma^{M-1}(P\sum_i\frac{\eta_i}{p_i}-(2a-1)P))}(n)\,\mathrm{e}^{\pi\mathrm{i}\frac{n^2}{2PN}}\,B_{2m+1}\left(\frac{n}{2PN}\right)\Bigg\}\end{aligned}$$

With this term, the mean value zero condition is satisfied even in this case, and we can apply the result of Coro. 11 to obtain the exact asymptotic expansion as required. □

We have thus obtained that the WRT invariant $Z_{N-2}(\mathcal{M})$ for the M-exceptional fibered Seifert homology sphere $\mathcal{M}=\Sigma(\vec{p})$ is written in a limit $N\to\infty$ as a sum of exponentially divergent terms and a *tail* part;

$$Z_{N-2}(\mathcal{M}) \simeq \sum_{k=0}^{M-3} N^{M-3-k}\, Z_{N-2}^{(k)}(\mathcal{M}) + \text{a tail part} \tag{4.25}$$

Here a *tail part* means an infinite power series of $1/N$, and it corresponds to a contribution from the trivial connection. Among the divergent terms in (4.25), the dominating term in the limit $N\to\infty$ is $Z_{N-2}^{(0)}(\mathcal{M})$, which is read as follows.

Corollary 17. *In a limit $N\to\infty$, the asymptotics of the WRT invariant $\tau_N(\mathcal{M})$ for the M-exceptional fibered Seifert homology sphere $\mathcal{M}=\Sigma(\vec{p})$ is dominated by $N^{M-3}\cdot Z_{N-2}^{(0)}(\mathcal{M})$, namely*

$$\begin{aligned}Z_{N-2}(\mathcal{M}) \sim N^{M-3}\cdot \mathrm{e}^{-\frac{\phi(\vec{p})}{2N}\pi\mathrm{i}}\,\frac{\mathrm{i}^{\mathrm{m}_2(M)-1}\,\mathrm{e}^{-\frac{3}{4}\pi\mathrm{i}}}{2\sqrt{2}\,(M-2)!} \\ \times\sum_{\vec{\ell}}\mathbf{S}_{\vec{\ell}}^{\sigma_1^{M-1}(\vec{E})}\,C_{\vec{p}}\left(\vec{\ell}\right)\,\mathrm{e}^{-\frac{P}{2}\left(1+\sum_{j=1}^{M}\frac{\ell_j}{p_j}\right)^2\pi\mathrm{i}N}.\end{aligned} \tag{4.26}$$

i.e.,

$$Z_{N-2}(\mathcal{M}) \sim N^{M-3} \frac{2^{M-2}}{(M-2)!\ \sqrt{P}} \mathrm{e}^{-\frac{\phi(\vec{p})}{2N}\pi\mathrm{i}} \mathrm{e}^{-\frac{2M+1}{4}\pi\mathrm{i}} \sum_{\vec{\ell}} C_{\vec{p}}\left(\vec{\ell}\right) \mathrm{e}^{-\frac{P}{2}\left(1+\sum_{j=1}^{M}\frac{\ell_j}{p_j}\right)^2\pi\mathrm{i}N}$$
$$\times(-1)^{MP\left(1+\sum_j\frac{\ell_j}{p_j}\right)+P\sum_j\frac{1}{p_j}+P\sum_j\sum_{k\neq j}\frac{\ell_k}{p_j p_k}} \left[\prod_{j=1}^{M}\sin\left(P\,\frac{\ell_j}{p_j^2}\,\pi\right)\right] \qquad (4.27)$$

where the sum of M-tuples $\vec{\ell}$ runs over D-dimensional space (3.7)*, and the function $C_{\vec{p}}\left(\vec{\ell}\right)$ is defined by*

$$C_{\vec{p}}\left(\vec{\ell}\right) = \sum_{n=1}^{2P}\chi_{2P}^{\vec{\ell}}(n)\,B_{M-2}\left(\frac{n}{2P}\right) \qquad (4.28)$$

We note that the invariance (3.6) of the periodic function $\chi_{2P}^{\vec{\ell}}(n)$ indicates that

$$C_{\vec{p}}\left(\sigma_i\sigma_j(\vec{\ell})\right) = C_{\vec{p}}\left(\vec{\ell}\right) \qquad (4.29)$$

By construction, the asymptotics (4.26) should coincide with (2.19) which follows from residue part of (2.16). We do not have a direct proof, and we have checked the equivalence numerically for several $\vec{p}$'s. Recalling the path integral approach, the sum of M-tuple $\vec{\ell}$ can be regarded as a label of the gauge equivalent class of flat connections α in (1.4), and we can identify the Chern–Simons invariant with

$$\mathrm{CS}\left(A_{\alpha(\vec{\ell})}\right) = -\frac{P}{4}\left(1+\sum_{j=1}^{M}\frac{\ell_j}{p_j}\right)^2 \mod 1 \qquad (4.30)$$

See Refs. 10, 28 for computations of the Chern–Simons invariant for the Seifert homology spheres. Note that this value originates from the T-matrix (3.16) of the vector modular form. Correspondingly, the Reidemeister torsion is given by

$$\sqrt{T_{\alpha(\ell)}} = \left|\prod_{j=1}^{M}\sin\left(P\,\frac{\ell_j}{p_j^2}\,\pi\right)\right|\cdot C_{\vec{p}}\left(\vec{\ell}\right) \qquad (4.31)$$

Here the product of sin-functions originates from the S-matrix of the vector modular form.

4.2 Ohtsuki Series

A tail part in the asymptotic formula (4.22) has a simple generating function as was studied in (2.17).

Theorem 18. *Let the T-series be defined by* (4.24)*. Then the generating function of the T-series is*

$$\frac{\prod_{j=1}^{M} \sinh\left(\frac{P}{p_j}\, x\right)}{[\sinh(Px)]^{M-2}} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{T_{\vec{p}}(k)}{(2\,k)!}\, x^{2k} \tag{4.32}$$

Proof. We first study a case of $\sum_j \frac{1}{p_j} < 1$. Using (4.10), we have

$$\frac{\prod_{j=1}^{M} \sinh\left(\frac{P}{p_j}\, x\right)}{[\sinh(Px)]^{M-2}} = \frac{(-1)^{M-1}}{2} \sum_{k=0}^{\infty} \sum_{n=0}^{2P-1} \binom{k+M-3}{k} \chi_{2P}^{\vec{E}}(n)\, \mathrm{e}^{-\left(n+2P\left(k+\frac{M-3}{2}\right)\right)x}$$

We equate this expression with $\sum_{k=0}^{\infty} \frac{T_k}{(2\,k)!}\, x^{2k}$. Applying the Mellin transformation, we get

$$T_k = \frac{(-1)^M}{2}\, \frac{(2\,P)^{2k}}{(M-3)!} \sum_{n=0}^{2P-1} \chi_{2P}^{\vec{E}}(n) \sum_{j=0}^{M-3} \frac{1}{1+j+2\,k}\, B_{1+j+2k}\left(\frac{n+P\,(M-3)}{2\,P}\right) \times \sum_{k=j}^{M-3} S_{M-3}^{(k)} \binom{k}{j} \left(\frac{P\,(M-3)-n}{2\,P}\right)^{k-j}$$

Here we have used (3.33). Identities (4.14) and (4.19) give

$$\sum_{k=j}^{M-3} S_{M-3}^{(k)} \binom{k}{j} \left(\frac{P\,(M-3)-n}{2\,P}\right)^{k-j} = (-1)^{M+1+j}\,(j+1)\, f_{j+1}^{M-2}\left(\frac{n}{2\,P}-\frac{1}{2}\right)$$

then we have

$$T_k = \frac{1}{2}\, \frac{(2\,P)^{2k}}{(M-3)!} \sum_{n=0}^{2P-1} \chi_{2P}^{\vec{E}}(n) \times \sum_{j=1}^{M-2} (-1)^j\, \frac{j}{j+2\,k}\, f_j^{M-2}\left(\frac{n}{2\,P}-\frac{1}{2}\right) B_{2k+j}\left(\frac{n+P\,(M-3)}{2\,P}\right) \tag{4.33}$$

For this expression, we substitute an identity

$$B_{2k+j}\left(\frac{n+P\,(M-3)}{2\,P}\right) = B_{2k+j}\left(\frac{n}{2\,P}+\frac{\mathrm{m}_2(M)-1}{2}\right) + (2\,k+j) \sum_{m=0}^{\lfloor \frac{M-4}{2} \rfloor} \left(\frac{n}{2\,P}+m+\frac{\mathrm{m}_2(M)-1}{2}\right)^{2k+j-1}$$

which follows from (A.10). As we have

$$\sum_{j=1}^{M-2} j\, f_j^{M-2}\left(\frac{n}{2P}-\frac{1}{2}\right)\left(-\frac{n}{2P}-m-\frac{\mathrm{m}_2(M)-1}{2}\right)^{j-1} = F_{M-2}\left(\frac{n}{2P}-\frac{1}{2},-\frac{n}{2P}-m-\frac{\mathrm{m}_2(M)-1}{2}\right)=0$$

from (4.17), we obtain (4.32).

When $\sum_j \frac{1}{p_j} > 1$, we have another term coming from the second term in (4.12). This gives an additional term to (4.33);

$$T_k^{\mathrm{add}} = \frac{(-1)^M}{2}\,\frac{(2P)^{2k}}{(M-4)!}\sum_{a=1}^{\infty}\sum_{b=0}^{a-1}\sum_{\vec{\eta}(a)}{}' \sum_{j=1}^{M-3}(-1)^j \frac{j}{2k+j}\, f_j^{M-3}\left(\frac{1}{2}\sum_i \frac{\eta_i}{p_i}-b-\frac{1}{2}\right) \times\left(B_{2k+j}\left(\frac{M-4}{2}+\frac{1}{2}\sum_i\frac{\eta_i}{p_i}-b\right)+(-1)^j B_{2k+j}\left(\frac{M-2}{2}-\frac{1}{2}\sum_i\frac{\eta_i}{p_i}+b\right)\right)$$

Recalling Lemma 14 and applying the same method with above, we recover (4.24). □

We give explicit forms of some T-series as follows;

$$T_{\vec{p}}(0)=0 \tag{4.34}$$

$$T_{\vec{p}}(1)=4P \tag{4.35}$$

$$T_{\vec{p}}(2)=8P^3\left(2-M+\sum_{j=1}^{M}\frac{1}{p_j^{\,2}}\right) \tag{4.36}$$

$$T_{\vec{p}}(3)=4P^5\left(5\left(\sum_{j=1}^{M}\frac{1}{p_j^{\,2}}+2-M\right)^2-2\left(\sum_{j=1}^{M}\frac{1}{p_j^{\,4}}+2-M\right)\right) \tag{4.37}$$

Based on the exact asymptotic expansion (4.22) of the WRT invariant, we extract the *tail* part and define the formal q-series $\tau_\infty(\mathcal{M})$ as a quantum invariant of the Seifert homology sphere $\mathcal{M}=\Sigma(\vec{p})$ by identifying $\exp(2\pi\,\mathrm{i}/N)$ with q;

$$q^{\frac{\phi(\vec{p})}{4}-\frac{1}{2}}\,(q-1)\cdot\tau_\infty(\mathcal{M})=\sum_{k=0}^{\infty}\frac{T_{\vec{p}}(k)}{k!}\left(\frac{\log q}{4P}\right)^k \tag{4.38}$$

This invariant $\tau_\infty(\mathcal{M})$ of the formal q-series coincides with the invariant $\tau_N^{\mathrm{int}}(\mathcal{M})$ defined in (2.16). Namely we have an integral expression for $\tau_\infty(\mathcal{M})$.

The Ohtsuki series [40] $\lambda_n(\mathcal{M})$ is defined from the formal q-series $\tau_\infty(\mathcal{M})$ by

$$\tau_\infty(\mathcal{M}) = \sum_{n=0}^{\infty} \lambda_n(\mathcal{M})\,(q-1)^n \tag{4.39}$$

Then the Ohtsuki series $\lambda_n(\mathcal{M})$ for $\mathcal{M} = \Sigma(\vec{p})$ is computed as follows.

Theorem 19. *The Ohtsuki series $\lambda_n(\mathcal{M})$ for the Seifert homology sphere $\mathcal{M} = \Sigma(\vec{p})$ is written in terms of $\vec{p}$ and $\phi(\vec{p})$ defined in* (2.11) *as*

$$\lambda_n(\mathcal{M}) = \frac{2}{(n+1)!}\left[\prod_{j=0}^{n}\left(\frac{P}{4}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2} - \frac{\phi(\vec{p})}{4} - j\right)\right] G(x)\bigg|_{x=0} \tag{4.40}$$

where the function $G(x)$ is

$$G(x) = \frac{\prod_{j=1}^{M}\sinh\left(\frac{x}{p_j}\right)}{[\sinh(x)]^{M-2}} \tag{4.41}$$

Proof. From (4.38) and (A.15), we have

$$(q-1)\,\tau_\infty(\mathcal{M}) = \sum_{m=0}^{\infty}\binom{\frac{1}{2}-\frac{\phi(\vec{p})}{4}}{m}(q-1)^m \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{S_j^{(k)}}{j!}\frac{T_{\vec{p}}(k)}{(4\,P)^k}(q-1)^j$$

We then have

$$\begin{aligned}\lambda(\mathcal{M}) &= \sum_{j=0}^{n}\frac{1}{(j+1)!}\binom{\frac{1}{2}-\frac{\phi(\vec{p})}{4}}{n-j}\sum_{k=1}^{j+1} S_{j+1}^{(k)}\,\frac{T_{\vec{p}}(k)}{(4\,P)^k}\\ &= \frac{1}{(n+1)!}\sum_{m=0}^{n}\sum_{k=1}^{n+1-m}\binom{m+k}{m} S_{n+1}^{(m+k)}\left(\frac{1}{2}-\frac{\phi(\vec{p})}{4}\right)^m\frac{T_{\vec{p}}(k)}{(4\,P)^k}\end{aligned}$$

where in the second equality we have expanded the binomial coefficient in terms of the Stirling number of the first kind using (A.14), then we have applied (A.17). Theorem 19 shows that the function $G(x)$ (4.41) gives

$$\left(P\,\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2k} G(x)\bigg|_{x=0} = \frac{1}{2}\,T_{\vec{p}}(k)$$

Substituting this expression and recalling (A.14), we obtain the required formula. $\square$

Explicit forms of the lowest 3 Ohtsuki series $\lambda_n(\mathcal{M})$ for $\mathcal{M} = \Sigma(\vec{p})$ are

$$\lambda_0(\mathcal{M}) = 1 \tag{4.42}$$

$$\lambda_1(\mathcal{M}) = 6\,\lambda_C(\mathcal{M}) \tag{4.43}$$

$$\begin{aligned}\lambda_2(\mathcal{M}) = \frac{3\,(\phi(\vec{p}))^2 + 12\,\phi(\vec{p}) - 4}{96} - \frac{P}{16}\left(2 - M + \sum_{j=1}^{M}\frac{1}{{p_j}^2}\right)(\phi(\vec{p}) + 2) \\ + \frac{P^2}{96}\left(5\left(2 - M + \sum_{j=1}^{M}\frac{1}{{p_j}^2}\right)^2 - 2\left(2 - M + \sum_{j=1}^{M}\frac{1}{{p_j}^4}\right)\right)\end{aligned} \tag{4.44}$$

where $\lambda_C(\mathcal{M})$ is the Casson invariant of the Seifert homology sphere $\mathcal{M} = \Sigma(\vec{p})$ [12, 39]

$$\lambda_C(\mathcal{M}) = -\frac{1}{8} + \frac{1}{24\,P}\left(1 + \sum_{k=1}^{M}\left(\frac{P}{p_k}\right)^2 - (M-2)\,P^2\right) - \frac{1}{2}\sum_{k=1}^{M} s\left(\frac{P}{p_k}, p_k\right) \tag{4.45}$$

The relationship between the Casson invariant $\lambda_C(\mathcal{M})$ and the Ohtsuki series $\lambda_1(\mathcal{M})$ was first proved in Ref. 38. See Ref. 53 for a computation of $\lambda_2(\mathcal{M})$.

4.3 Lattice Points

We have seen that the asymptotics of the WRT invariant is dominated by the term (4.26), which shows that the number of terms in the sum of $\vec{\ell}$ are at most D defined in (3.7). Though, as was studied in Refs. 19, 20 for cases of $M = 3$ and $M = 4$, the function $C_{\vec{p}}(\vec{\ell})$ may vanish for some $\vec{\ell}$'s.

Theorem 20. *We fix M-tuple $\vec{p}$ with pairwise coprime positive integers p_j, and let the function $C_{\vec{p}}(\vec{\ell})$ be defined by* (4.28) *for $\vec{\ell} \in \mathbb{Z}^M$ satisfying $1 \leq \ell_j \leq p_j - 1$. Due to* (4.29), *we have D independent functions. We set $\gamma(\vec{p})$ as the number of M-tuples $\vec{\ell}$ satisfying*

$$C_{\vec{p}}(\vec{\ell}) \neq 0,$$

and $L(\vec{p})$ as the integral lattice points $\vec{\ell} \in \mathbb{Z}^M_{>0}$ inside the M-dimensional tetrahedron,

$$0 < \sum_{j=1}^{M}\frac{\ell_j}{p_j} < 1$$

Then we have

$$D - \gamma(\vec{p}) \geq L(\vec{p}) \tag{4.46}$$

Proof. As a generalization of the function $C_{\vec{p}}(\vec{\ell})$ defined in (4.28), we define $C^k_{\vec{p}}(\vec{\ell})$ for $k \geq 0$ by

$$C^k_{\vec{p}}(\vec{\ell}) = \sum_{n=1}^{2P}\chi^{\vec{\ell}}_{2P}(n)\,B_k\left(\frac{n}{2\,P}\right) \tag{4.47}$$

in terms of the Bernoulli polynomials. We have $C_{\vec{p}}(\vec{\ell}) = C_{\vec{p}}^{M-2}(\vec{\ell})$. As a generating function $Z_{\vec{p}}^{\vec{\ell}}(t)$ of these polynomials, we define

$$Z_{\vec{p}}^{\vec{\ell}}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} C_{\vec{p}}^k(\vec{\ell}), \tag{4.48}$$

Using (A.8), we have

$$Z_{\vec{p}}^{\vec{\ell}}(t) = \frac{t}{\mathrm{e}^t - 1} \sum_{n=1}^{2P} \chi_{2P}^{\vec{\ell}}(n)\, \mathrm{e}^{\frac{t}{2P} n} \tag{4.49}$$

In the case of $0 < \sum_j \frac{\ell_j}{p_j} < 1$, we have

$$\sum_{n=0}^{2P} \chi_{2P}^{\vec{\ell}}(n)\, z^n = -z^P \prod_{j=1}^{M} \left(z^{P \frac{\ell_j}{p_j}} - z^{-P \frac{\ell_j}{p_j}} \right) \tag{4.50}$$

which gives

$$Z_{\vec{p}}^{\vec{\ell}}(t) = -\frac{t}{\mathrm{e}^{\frac{t}{2}} - \mathrm{e}^{-\frac{t}{2}}} \prod_{j=1}^{M} \left(\mathrm{e}^{\frac{\ell_j}{2p_j} t} - \mathrm{e}^{-\frac{\ell_j}{2p_j} t} \right).$$

This shows that

$$Z_{\vec{p}}^{\vec{\ell}}(t) = -\left(\prod_{j=1}^{M} \frac{\ell_j}{p_j} \right) t^M + O(t^{M+2})$$

and that $C_{\vec{p}}^k(\vec{\ell}) = 0$ for $0 \leq k \leq M-1$. So we have $C_{\vec{p}}(\vec{\ell}) = 0$ when $0 < \sum_j \frac{\ell_j}{p_j} < 1$.

The invariance (3.6) of the periodic functions $\chi_{2P}^{\vec{\ell}}(n)$ proves the statement of the theorem. □

In the case of $\sum_j \frac{\ell_j}{p_j} > 1$, the generating function (4.50) should be replaced with a formula like (4.12), and we do not know whether the function $C_{\vec{p}}(\vec{\ell})$ vanishes. We conjecture that, when $\sum_j \frac{\ell_j}{p_j} > 1$, we have $C_{\vec{p}}(\ell) = 0$ iff $\chi_{2P}^{\vec{\ell}}(n)$ coincides with $\chi_{2P}^{\vec{\ell}'}(n)$ s.t. $\sum_j \frac{\ell'_j}{p_j} < 1$.

Conjecture 1. *Under the conditions of Theorem 20, we have*

$$D - \gamma(\vec{p}) = L(\vec{p}) \tag{4.51}$$

This conjecture was proved for $M \leq 4$ in Refs. 19, 20. It states that the number of the flat connections which contribute as (4.26) coincides with the number of integral lattice points inside the M-dimensional tetrahedron.

4.4 Ehrhart Polynomial

Explicit form of the number $L(\vec{p})$ of the lattice points inside the M-dimensional tetrahedron was first computed by Mordell [37] for cases of $M = 3$ and $M = 4$;

- $M = 3$;

$$L(p_1, p_2, p_3) = \frac{1}{4}(p_1 - 1)(p_2 - 1)(p_3 - 1) + \frac{1}{12P} - \frac{1}{4} - \frac{P}{12}\left(1 - \frac{1}{p_1^2} - \frac{1}{p_2^2} - \frac{1}{p_3^2}\right) - s(p_1 p_2, p_3) - s(p_2 p_3, p_1) - s(p_1 p_3, p_2) \tag{4.52}$$

- $M = 4$,

$$L(p_1, p_2, p_3, p_4) = \frac{1}{8}\prod_{j=1}^{4}(p_j - 1) + \frac{3}{8} - \frac{P}{12} + \frac{P}{24}\sum_{j=1}^{4}\frac{1 + p_j}{p_j^2} + \frac{1}{24P}\left(1 - \sum_{j=1}^{4} p_j\right) - \frac{P}{24}\sum_{j \neq k}^{4}\frac{1}{p_j^2 p_k} - \frac{1}{2}\sum_{j=1}^{4} s\left(\frac{P}{p_j}, p_j\right) + \frac{1}{2}\sum_{j \neq k}^{4} s\left(\frac{P}{p_j p_k}, p_j\right) \tag{4.53}$$

Here $s(b, a)$ is the Dedekind sum (A.1). For higher dimension M, the lattice points $L(\vec{p})$ might be written in terms of Zagier's higher-dimensional Dedekind sum [59], but there seems to exist no applicable expressions.

Although, there is a useful tool to count the lattice points (see, *e.g.*, Ref. 6). Let $\mathcal{P}$ be the M-dimensional open tetrahedron with integer vertices, $(p_1, 0, \ldots, 0)$, $(0, p_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, p_M)$, and $(0, \ldots, 0)$;

$$\mathcal{P} = \left\{(\ell_1, \ldots, \ell_M) \in \mathbb{Z}^M \;\middle|\; \sum_{j=1}^{M}\frac{\ell_j}{p_j} < 1, \ell_k > 0\right\} \tag{4.54}$$

Let $E_{\mathcal{P}}(t)$ denote the number of lattice points in the dilated tetrahedron $t\mathcal{P}$. So we have

$$L(\vec{p}) = E_{\mathcal{P}}(t = 1)$$

In the same manner, we suppose that $E_{\overline{\mathcal{P}}}(t)$ denotes the number of lattice points of the closure of $t\mathcal{P}$,

$$E_{\overline{\mathcal{P}}}(t) = \#\left\{(m_1, \ldots, m_M) \in \mathbb{Z}^M \mid \sum_{j=1}^{M}\frac{m_j}{p_j} \leq t, m_k \geq 0\right\}$$

These functions, $E_{\mathcal{P}}(t)$ and $E_{\overline{\mathcal{P}}}(t)$, become polynomials of t [9], which are called the Ehrhart polynomial. Moreover we have the Ehrhart–Macdonald reciprocity formula [9, 33],

$$E_{\mathcal{P}}(-t) = (-1)^M E_{\overline{\mathcal{P}}}(t) \tag{4.55}$$

In general, the number of lattice points $E_{\overline{\mathcal{P}}}(t)$ becomes polynomial of t for arbitrary polytope $\overline{\mathcal{P}}$ [9]. We set the coefficients of the Ehrhart polynomial as

$$E_{\overline{\mathcal{P}}}(t) = c_M(\mathcal{P})\, t^M + c_{M-1}(\mathcal{P})\, t^{M-1} + \cdots + c_0(\mathcal{P}) \tag{4.56}$$

It is well known that $c_M(\mathcal{P})$ is the volume of $\mathcal{P}$, $c_M(\mathcal{P}) = \mathrm{Vol}(\mathcal{P})$, $c_{M-1}(\mathcal{P})$ is a half of the boundary surface area, $c_{M-1}(\mathcal{P}) = \frac{1}{2}\,\mathrm{Vol}(\partial\mathcal{P})$. The coefficient $c_0(\mathcal{P})$ is the Euler characteristic $\chi(\mathcal{P})$, and $c_0(\mathcal{P}) = 1$ when $\mathcal{P}$ is the convex polytope.

The first nontrivial coefficient of the Ehrhart polynomial for the M-dimensional tetrahedron is thus $c_{M-2}(\mathcal{P})$. In our case of the M-dimensional tetrahedron with pairwise coprime integers p_j, we have [5, 8]

$$\begin{aligned}&(M-2)! \cdot c_{M-2}(\mathcal{P}) \\ &\qquad = \frac{M}{4} + \frac{1}{24\,P}\left(2 - \sum_{k=1}^{M}\left(\frac{P}{p_k}\right)^2 + 3\left(\sum_{k=1}^{M}\frac{P}{p_k}\right)^2\right) - \sum_{k=1}^{M} s\left(\frac{P}{p_k}, p_k\right)\end{aligned} \tag{4.57}$$

Recalling the Casson invariant $\lambda_C(\mathcal{M})$ (4.45) for the Seifert homology sphere $\mathcal{M} = \Sigma(\vec{p})$, which is proportional to the first Ohtsuki series (4.43), we have the following.

Proposition 21. *The Casson invariant $\lambda_C(\mathcal{M})$ for $\mathcal{M} = \Sigma(\vec{p})$ is related to the first nontrivial coefficient of the Ehrhart polynomial for the M-dimensional tetrahedron $\mathcal{P}$* (4.54)*;*

$$\lambda_C(\mathcal{M}) - \frac{(M-2)!}{2}\, c_{M-2}(\mathcal{P}) = -\frac{M+1}{8} + \frac{M-2}{24}\, P - \frac{P}{8} \sum_{1\leq j<k\leq M} \frac{1}{p_j\, p_k} \tag{4.58}$$

It is remarked that the residue formula for $c_{M-2}(\mathcal{P})$ given in Ref. 5 looks like an expression (2.10) of the WRT invariant $\tau_N(\mathcal{M})$ for $\mathcal{M} = \Sigma(\vec{p})$.

5. Some Examples of Numerical Experiments

We give some numerical experiments on the asymptotic behavior of the WRT invariants for the Seifert homology spheres.

5.1 $\Sigma(2, 3, 5, 7, 11)$

For $\vec{p} = (2, 3, 5, 7, 11)$, we have $P = 2310$, $D = 30$, and $\phi(\vec{p}) = \frac{34189}{2310}$. The bases for 30-dimensional space is given by $\vec{\ell} = (1, 1, \ell_3, \ell_4, \ell_5)$ with $1 \leq \ell_3 \leq 2$, $1 \leq \ell_4 \leq 3$, $1 \leq \ell_5 \leq 5$. For all these 5-tuples $\vec{\ell}$, we can check that $C_{\vec{p}}(\vec{\ell}) \neq 0$, which supports Conjecture 1 as we have $\sum_j \frac{1}{p_j} = \frac{2927}{2310} > 1$.

N	exact result for Z_N	asymptotics $Z_N^{(0)}$
22	$-13.346013 + 17.397906\,\mathrm{i}$	$-12.2403 + 16.7013\,\mathrm{i}$
23	$-0.57682556 - 0.51108147\,\mathrm{i}$	$0.020572 + 0.004140\,\mathrm{i}$
98	$0.93263590 - 0.49655457\,\mathrm{i}$	$0.323366 + 0.0057023\,\mathrm{i}$
99	$22.826764 - 367.89360\,\mathrm{i}$	$22.8460 - 365.870\,\mathrm{i}$
100	$464.33437 - 287.59556\,\mathrm{i}$	$475.688 - 287.973\,\mathrm{i}$
998	$9.2292110 - 9.3324129\,\mathrm{i}$	$10.7013 - 1.60581\,\mathrm{i}$
999	$-52995.123 - 87204.076\,\mathrm{i}$	$-53072.7 - 87187.8\,\mathrm{i}$
1000	$694.74344 + 9181.2935\,\mathrm{i}$	$683.369 + 9183.49\,\mathrm{i}$
2398	$-64.891808 + 46.620794\,\mathrm{i}$	$-62.4971 + 47.9275\,\mathrm{i}$
2399	$320910.08 + 27551.395\,\mathrm{i}$	$321128. + 27510.1\,\mathrm{i}$
2400	$142206.21 - 1871.8080\,\mathrm{i}$	$142145. - 1869.06\,\mathrm{i}$
2401	$214250.48 - 80025.187\,\mathrm{i}$	$214270. - 79907.4\,\mathrm{i}$

Table 1: The WRT invariant $Z_N(\mathcal{M})$ for $\mathcal{M} = \Sigma(2,3,5,7,11)$. Asymptotic formula for $Z_N^{(0)}(\mathcal{M})$ is from (4.26).

In table 1 we give numerical results on the Witten invariant $Z_N(\mathcal{M})$ for $\mathcal{M} = \Sigma(2,3,5,7,11)$, which is performed with a help of PARI/GP. We give both the exact value $Z_N(\mathcal{M})$ and asymptotic value $Z_N^{(0)}(\mathcal{M})$ for several Ns. They vary much with N, and comparing these data we see an agreement.

5.2 $\Sigma(3,7,8,11,13,17)$

For $\vec{p} = (3,7,8,11,13,17)$, we have $\phi(\vec{p}) = \frac{338099}{408408}$ and $D = 5040$. This $D = 5040$-dimensional vector space is spanned by $\vec{\ell} = (1, 1 \leq \ell_2 \leq 3, 1 \leq \ell_3 \leq 7, 1 \leq \ell_4 \leq 5, 1 \leq \ell_5 \leq 6, 1 \leq \ell_6 \leq 8)$. Among these $D = 5040$ bases, we check that $C_{\vec{p}}(\vec{\ell}) = 0$ when $\vec{\ell} = (1,1,1,1,1,1)$, $(1,1,1,1,1,2)$, $(1,1,1,1,1,3)$, $(1,1,1,1,2,1)$, $(1,1,1,1,2,2)$, $(1,1,1,1,3,1)$, $(1,1,1,2,1,1)$, $(1,1,1,2,1,2)$, $(1,1,1,2,2,1)$, $(1,1,2,1,1,1)$, $(1,2,1,1,1,1)$, which supports Conjecture 1.

Numerical results on the exact value and the asymptotics of the Witten invariant for $\Sigma(\vec{p})$ are summarized in Table 2. We see an agreement.

6. Conclusion and Discussion

We have studied the asymptotic expansion of the SU(2) WRT invariant $\tau_N(\mathcal{M})$ for the M-exceptional fibered Seifert homology spheres $\mathcal{M} = \Sigma(\vec{p})$ in $N \to \infty$ number theoretically. We have found that the invariant can be written in terms of a limiting value of *fractional* derivative, *i.e.* derivative of the Eichler integral, of the vector modular forms with weight $3/2$ and $1/2$. This supports a result [13] that the WRT invariant is a *limiting* value of the holomorphic function in a

N	exact value $Z_N(\mathcal{M})$	asymptotics $Z_N^{(0)}(\mathcal{M})$
58	365.32895 + 679.07006 i	351.149 + 691.982 i
59	1331.8460 − 433.95047 i	1358.51 − 437.953 i
60	−944.99493 + 765.34451 i	−915.949 + 742.606 i
61	130.91099 + 2814.5744 i	62.8489 + 2763.93 i
118	−0.8206017 + 61.590246 i	0.782372 + 60.1248 i
119	8.1857781 + 13.369868 i	0.0195662 + 0.0062675 i
120	5259.2853 + 4064.4029 i	5232.38 + 4043.94 i
121	8733.0140 + 5274.8273 i	8659.21 + 5338.15 i
238	−219.36738 − 1.608943 i	−216.499 + 1.53462 i
239	−6151.0562 − 5617.75586 i	−6220.64 − 5620.95 i
240	−11.492746 + 6.1192358 i	1.67454 + 2.34920 i
241	−26057.019 − 52201.108 i	−25950.5 − 52634.8 i
242	49736.853 − 46390.033 i	49818.0 − 46337.0 i
243	189895.62 + 265408.04 i	189029. + 265225. i
244	3782.8814 − 12474.142 i	3814.35 − 12433.5 i
998	21039.448 + 18091.568 i	21107.1 + 18191.2 i
999	−12.505553 + 49.861847 i	−0.0331549 + 0.0338852 i
1000	78229.306 − 164203.36 i	78333.1 − 164618. i

Table 2: The WRT invariant $Z_N(\mathcal{M})$ for $\mathcal{M} = \Sigma(3, 7, 8, 11, 13, 17)$. Asymptotic formula for $Z_N^{(0)}(\mathcal{M})$ is from (4.26).

limit that q tends to the N-th root of unity. By use of the nearly modular property of the Eichler integral, we have obtained an asymptotic expansion of $\tau_N(\mathcal{M})$ in the large N limit.

Although an asymptotic behavior of the WRT invariant was previously studied in, *e.g.*, Ref. 31, the correspondence between modular forms and the quantum invariants enables us to relate topological invariants such as the Chern–Simons invariants, the Reidemeister torsion, and the Casson invariant, with geometries of modular forms. For example, we have found that the number of the gauge equivalent classes of flat connections, which dominate the WRT invariant in the large-N limit, is related to the number of integral lattice points inside the M-dimensional tetrahedron. From this view, we have established that the Casson invariant for the Seifert homology sphere has a relationship with the first non-trivial coefficient of the Ehrhart polynomial.

In our previous papers [21, 22] we have shown that the WRT invariants for the Seifert manifolds with 3-exceptional fibers coincides with a limiting value of the Ramanujan mock theta functions. Investigated [23] is a modular transformation formula of the newly proposed mock theta functions based on explicit form of the WRT invariants. This intriguing correspondence seems to originate from a result that the integral expression (2.16) of the WRT invariant has a

connection with the Mordell integral [36]. Our results presented here will shed a new light on geometric and topological aspects of modular forms.

Though we have studied only the SU(2) invariant, the Witten partition function (1.1) can be defined for arbitrary gauge group, and an explicit form of the invariant for the Seifert manifold is given [15]. Extending the method of Lawrence and Rozansky, it is shown [35] that the partition function can be written in the integral form which can be interpreted as the matrix model, and that it becomes a sum of local contributions from the flat connections. This fact is recently reinterpreted from the viewpoint of the path integral by use of non-abelian localization [4]. As it is well known that the Chern–Simons perturbation theory of the SU(N) Witten invariant as a $1/N$ expansion can be interpreted from the string theory (see, *e.g.*, Ref. 34), it will be interesting to investigate the quantum invariants/modular forms correspondence for the WRT invariant associated with SU(N) gauge group as a generalization of the present work.

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A. Special Functions and Identities

A.1 Dedekind sum

The Dedekind sum is defined by (see, *e.g.*, Ref. 44)

$$s(b,a) = \sum_{k=1}^{a-1} \left(\left(\frac{k}{a}\right)\right) \left(\left(\frac{k\,b}{a}\right)\right) \tag{A.1}$$

where $((x))$ is the sawtooth function

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{when } x \notin \mathbb{Z} \\ 0 & \text{when } x \in \mathbb{Z} \end{cases}$$

The Dedekind sum can also be written as

$$s(b,a) = \frac{1}{4\,a} \sum_{k=1}^{a-1} \cot\left(\frac{k}{a}\,\pi\right) \cot\left(\frac{k\,b}{a}\,\pi\right) \tag{A.2}$$

The Dedekind sum is known to satisfy the reciprocity formula

$$s(b,a) + s(a,b) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{a\,b}\right) \tag{A.3}$$

We note

$$s(-b,a) = -s(b,a) \tag{A.4}$$

$$s(b,a) = s(c,a) \qquad \text{for } b\,c = 1 \mod a \tag{A.5}$$

A.2 Gauss sum

As a discrete analogue of the Gaussian integral, we have a formula of the Gauss sum as

$$\sum_{n=0}^{2N-1} \mathrm{e}^{-\frac{1}{2N} n^2 \pi \mathrm{i}} = \sqrt{2\,N}\,\mathrm{e}^{-\frac{1}{4}\pi \mathrm{i}} \tag{A.6}$$

The reciprocity formula of the Gauss sum follows from the Gauss integral as (see *e.g.* Refs. 7,26)

$$\sum_{n \mod N} \mathrm{e}^{\pi \mathrm{i} \frac{M}{N} n^2 + 2\pi \mathrm{i} k n} = \sqrt{\left|\frac{N}{M}\right|}\,\mathrm{e}^{\frac{\pi \mathrm{i}}{4}\,\mathrm{sign}(NM)} \sum_{n \mod M} \mathrm{e}^{-\pi \mathrm{i} \frac{N}{M}(n+k)^2} \tag{A.7}$$

where $N, M \in \mathbb{Z}$ and $N\,k \in \mathbb{Z}$, and $N\,M$ is even.

A.3 Bernoulli Polynomial

The n-th Bernoulli polynomial $B_n(x)$ is defined from the generating function as

$$\frac{t\,\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{k=0}^{\infty} B_k(x)\,\frac{t^k}{k!} \tag{A.8}$$

Some of them are written as follows;

$$B_0(x) = 1$$

$$B_1(x) = x - \frac{1}{2}$$

$$B_2(x) = x^2 - x + \frac{1}{6}$$

$$B_3(x) = x^3 - \frac{3}{2}\,x^2 + \frac{1}{2}\,x$$

These polynomials satisfy the following relations;

$$B_k(1-x) = (-1)^k\,B_k(x) \tag{A.9}$$

$$B_k(x+1) - B_k(x) = k\,x^{k-1} \tag{A.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} B_n(x) = n\,B_{n-1}(x) \tag{A.11}$$

Note that the Bernoulli function has the Fourier expansion as

$$B_k(x - \lfloor x \rfloor) = -k! \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{e^{2\pi i n x}}{(2\pi i n)^k} \tag{A.12}$$

and that

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x)\, y^{n-k} \tag{A.13}$$

A.4 Stirling Number

The Stirling number of the first kind $S_n^{(m)}$ denotes the (signed) number of permutations of n elements which contain m permutation cycles (see, *e.g.*, Ref. 57). The generating function of $S_n^{(m)}$ is written as

$$\prod_{j=0}^{n-1} (x - j) = \sum_{m=0}^{n} S_n^{(m)} x^m \tag{A.14}$$

and $S_n^{(m)} \neq 0$ when $n \geq m \geq 0$. Another form of the generating function is given by

$$\frac{(\log q)^m}{m!} = \sum_{n=m}^{\infty} S_n^{(m)} \frac{(q-1)^n}{n!} \tag{A.15}$$

Based on these generating functions, we have the recursion relations of $S_n^{(m)}$ as follows;

$$S_{n+1}^{(m)} = S_n^{(m-1)} - n\, S_n^{(m)} \tag{A.16}$$

$$\binom{m}{r} S_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} S_{n-k}^{(r)} S_k^{(m-r)} \tag{A.17}$$

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